M.C. Escher and maps between Elliptic Curves: A *very* expository talk on the work of Lenstra et. al.

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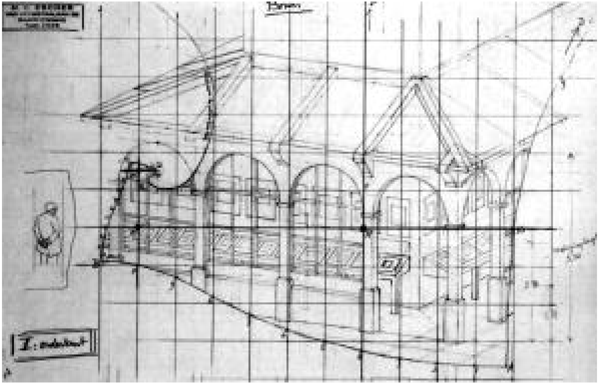
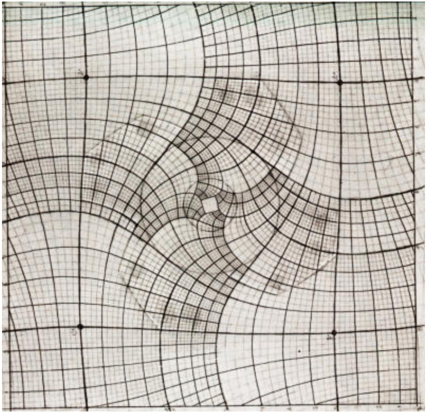
Prententententoonstelling ∼ "The Print Gallery" Escher - 1956

Escher's vision

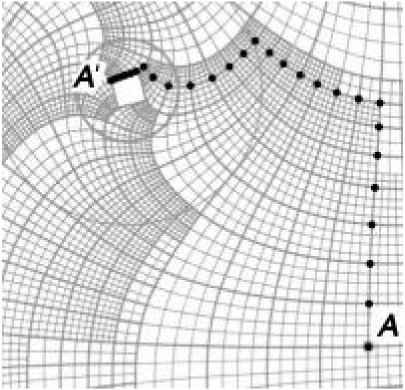
Escher had a dream of twisting a picture "from an annular bulge" and a "cyclic expansion . . . without beginning or end." In painstaking fashion he realized this dream by starting with simple drawings and deforming them according to a twisted grid he drew.



Escher's Grid

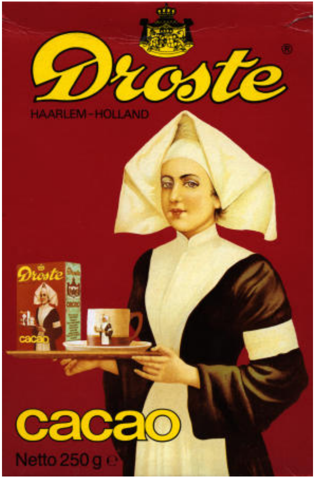


In 1999, Hendrik Lenstra noticed this Escher picture in a flight magazine and being a specialist in elliptic curves and lattices he noticed a familiar kind of symmetry in the picture: If you make 3 left turns at right angles(say following the rectangular window) you get to a place where the picture repeats.



Droste effect

He was able to see that after undoing Escher's transformation with his grid, there should be a "Droste effect" and set out to work with his former student Bart de Smit.



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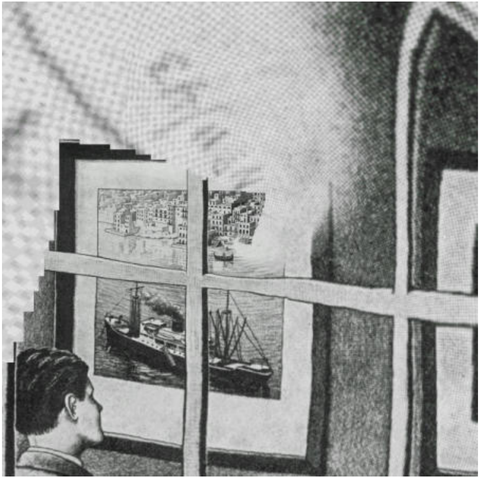

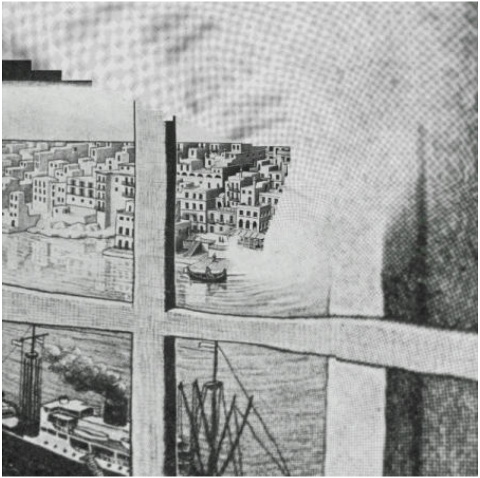

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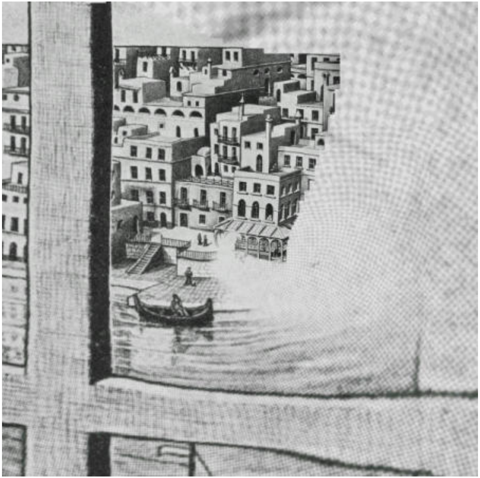

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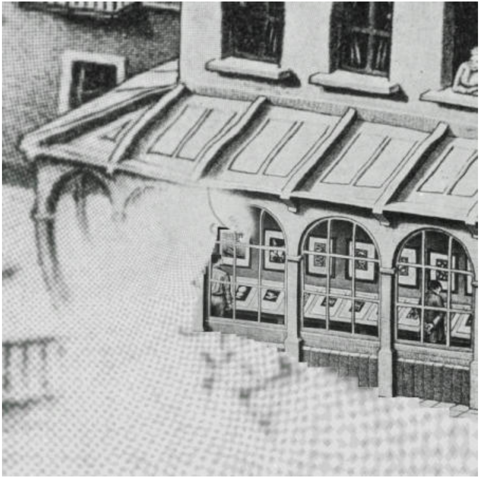

M.C. Escher and maps between Elliptic Curves

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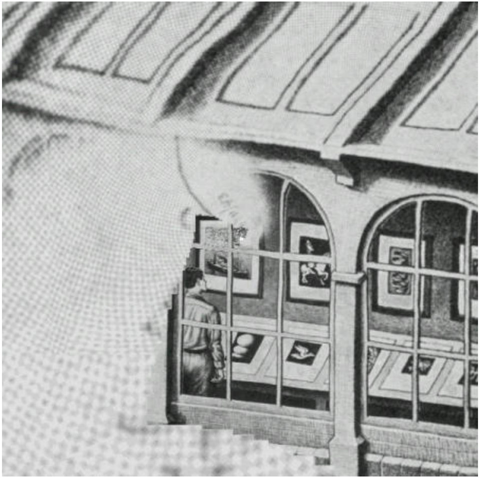

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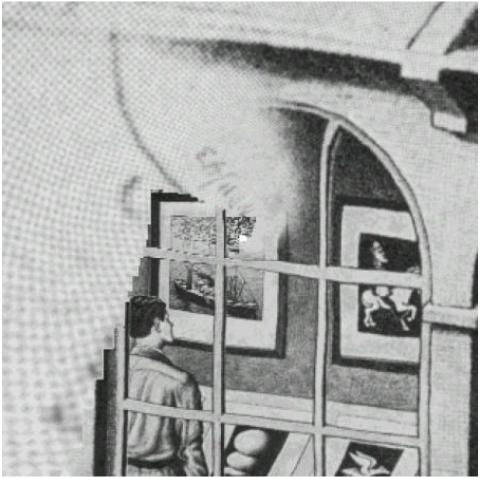

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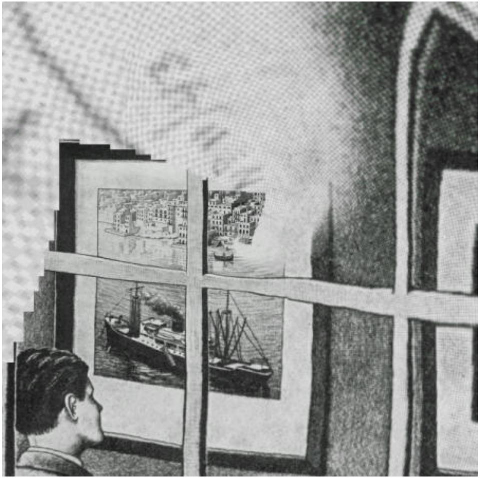


The Straightened Picture as a map

This straightened picture is then a self-similar one, wherein if we zoom in and double the picture 8 times, we recover the original! Thus the picture has multiplicative period $2^8 = 256$. We can make this rigorous by considering the real plane as the complex numbers so the straightened picture itself can be viewed as a function

$$\mathbb{C} \to \{\text{black}, \text{white}\}.$$

The center point we will soon see as a singularity, and when we remove it, this becomes a map

$$\mathbb{C}^\times/\langle 256\rangle \to \{\text{black}, \text{white}\}.$$

Escher's Picture as a map

The original picture can also be seen as (an approximation of) a map

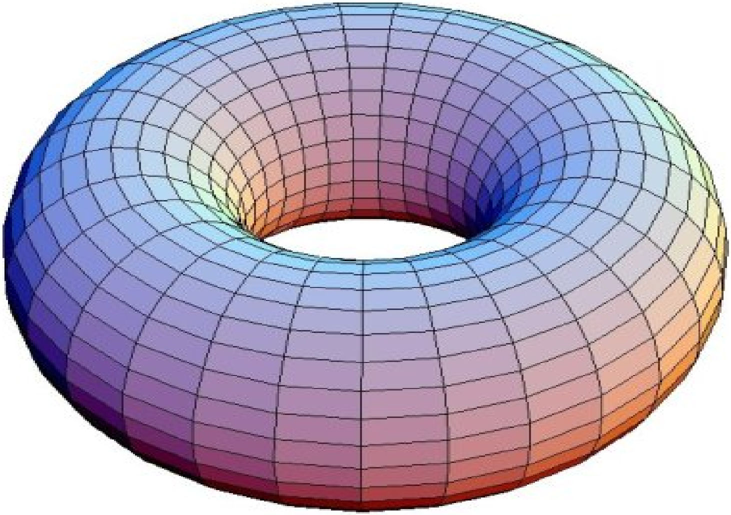
$$\mathbb{C}^\times \to \{\text{black}, \text{white}\}.$$

It will also have a period, but as we saw earlier, that period was both a dilation and a rotation, for now let's take for granted that the period has to be a nonzero complex number γ so that Escher's original picture is a map

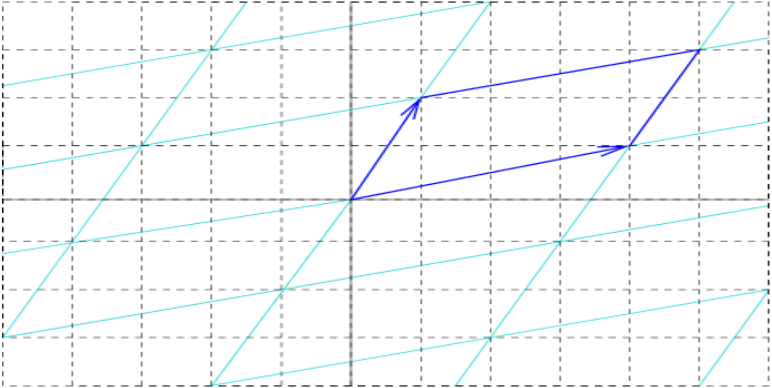
$$\mathbb{C}^\times / \langle \gamma \rangle \to \{\text{black}, \text{white}\}.$$

Moreover, Escher's grid may now be seen as giving a holomorphic isomorphism between $\mathbb{C}^\times / \langle \gamma \rangle$ and $\mathbb{C}^\times / \langle 256 \rangle$!

For us an elliptic curve over $\mathbb{C}$ is a torus



Unrolling a torus to the plane lets us recognize every elliptic curve over the complex numbers as $\mathbb{C}/L$ where $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ is a lattice in $\mathbb{C}$ with two independent generators $\omega_1, \omega_2 \in \mathbb{C}^\times$.



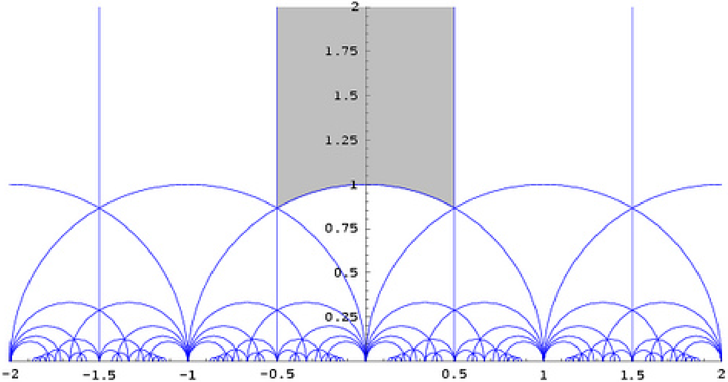
Isomorphisms

Now Escher's grid gave us a way to show that 2 objects, which we will see are actually elliptic curves, are the same. We need to consider when two elliptic curves are the same: we'll say so when there's an invertible map between the two of them. This means $\mathbb{C}/L_1$ and $\mathbb{C}/L_2$ are the same if there's an invertible map $\mathbb{C} \to \mathbb{C}$ sending L_1 to L_2.

An example of this might be just multiplication by a nonzero complex number. So through multiplication by $\pm 1/\omega_1$ we can limit ourselves to $L = \mathbb{Z} \oplus \tau\mathbb{Z}$ where $\tau = \frac{\pm\omega_2}{\omega_1}$ is in the upper half-plane.

In fact multiplication by nonzero complex numbers are exactly the invertible holomorphic maps between elliptic curves.

Then we can figure out which lattices give the same elliptic curve, i.e., those related by a invertible $\mathbb{Z}$-linear transformation that preserved the upper half plane. The complex numbers in the shaded region below $(SL_2(\mathbb{Z}) \setminus \mathcal{H})$ are then in one to one correspondence with the elliptic curves over $\mathbb{C}$.



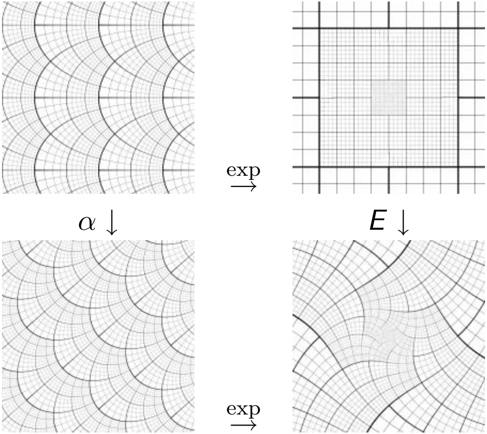
Therefore an elliptic curve over $\mathbb{C}$ can be written uniquely as $\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$. But then we can just apply the map $\exp(2\pi i\,\cdot)$, giving $\mathbb{C}^\times/\langle\gamma\rangle$ where $\gamma = \exp(2\pi i\tau)$!
Now we return to the case of Escher's Picture: we take 256 and

$$\gamma \approx \exp(3.1172277221 + 2.7510856371i).$$

Escher's Grid E is then given (before pulling back by the exponential) by a nonzero complex number α making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \to & L_{256} & \to & \mathbb{C} & \to & \mathbb{C}^\times/\langle 256\rangle & \to & 0 \\ & & \downarrow & & \alpha\downarrow & & E\downarrow & & \\ 0 & \to & L_\gamma & \to & \mathbb{C} & \to & \mathbb{C}^\times/\langle\gamma\rangle & \to & 0 \end{array}$$

Visually this is



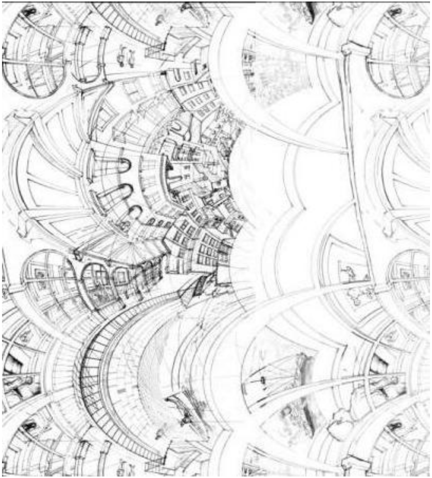
What about that hole?

We now see the center point as the singularity of the complex logarithm. Could Escher have understood that the center point is a singularity? Is that why he put a white hole in the center and signed his name there or did he simply realize he could be there forever drawing in details?

The Dutch team set out to reconstruct Escher's picture without a hole. The first step was to hire an artist, Hans Richter to reproduce a straightened picture, so they could feed it through the complex transformation.

The commissioned picture

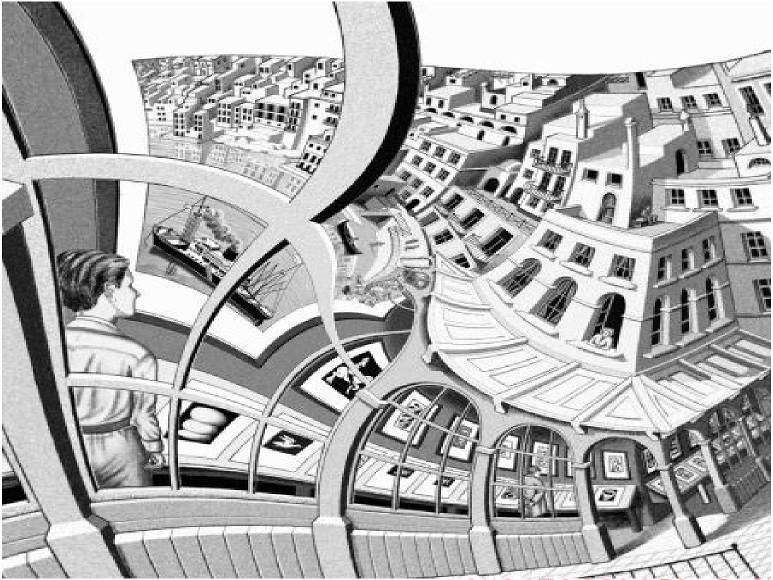
Now we pull back by the exponential



And fill in the greyscale

(This was done by another artist, Jacqueline Hofstra)

And then apply our exponential transformation!



First the standard Droste effect

And now the symmetry of Escher!

Thank you!

> *"It beats getting stoned, and it'll keep you off the streets"- Serge Lang (In the foreward to Math Talks for Undergraduates)*

For more on what the dutch team did, please see

http://escherdroste.math.leidenuniv.nl/