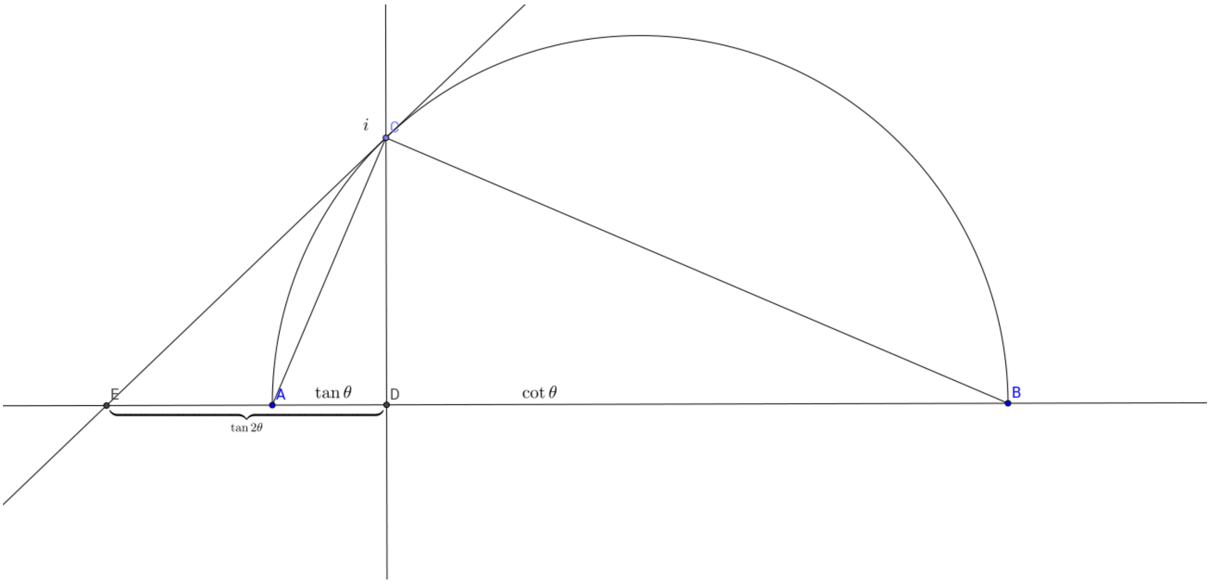
Hyperbolic Geometry Homework 2

Name: Answer Rubric

1. (12 points) Verify that the following picture describes the effect of k_θ on the imaginary axis when $0 < \theta < \pi/2$. That is, verify that all the lengths and intersection points are correct. What is the slope of the tangent line? What is the angle formed between the imaginary axis and its image under k_θ?



Solution: Since $0 < \theta < \pi/2$, $\tan\theta$ and $\cot\theta$ are both strictly positive. We know that the image of the imaginary axis is the circle with limit points $-\tan\theta$ and $\cot\theta$ by a previous exercise (**2 points**) (where we computed the circle for ANY element of $\mathrm{SL}_2(\mathbf{R})$.) More than that, we know the imaginary part of $k_\theta(it)$ is $y = \frac{t}{t^2\sin^2\theta+\cos^2\theta} = \frac{t}{(t^2-1)\sin^2\theta+1}$ (**2 points**) and the real part is $x = \cot\theta(1 - \frac{1}{(t^2-1)\sin^2\theta+1})$ (**2 points**).

Now we determine the tangent line at $t = 1$, i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 0$ and $y = 1$. We can readily compute that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{(-t^2-1)\sin^2\theta}{[(t^2-1)\sin^2\theta+1]^2},$$

and that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2t\cot\theta\sin^2\theta}{[(t^2-1)\sin^2\theta+1]^2} = \frac{t\sin(2\theta)}{[(t^2-1)\sin^2\theta+1]^2}.$$

Therefore by the chain rule, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(-t^2-1\sin^2\theta+1)}{t\sin(2\theta)}$. At $t = 1$ the numerator reduces to $-2\sin^2\theta + 1 = \cos^2\theta - \sin^2\theta = \cos(2\theta)$. Therefore the slope of our tangent line is $\cot(2\theta)$ (**4 points**). There are many ways to find this, e.g. by finding the equation of the circle and doing implicit differentiation or using some classical Euclidean trigonometry. Any way that is correct should receive all points. Take off no more than two points for algebra errors (and there must be distinct algebra errors in distinct steps to do so). Give at most 2 points if there is an error in the definition of a tangent line. Do not take away any points in this section if there was an error in finding x or y before any attempt to find the tangent line.

Once we know the tangent line has slope $\cot(2\theta)$, we know that the intercept with the real axis is at $-\tan(2\theta)$ (**1 points**). If we take the Euclidean right triangle formed by the real and imaginary axes and the tangent line, we see it has vertices $(0,0)$, $(-\tan 2\theta, 0)$, and $(0,1)$. We can read off that the angle between the imaginary axis and the tangent line has adjacent side length 1 and opposite side length $\tan(2\theta)$ so by vertical angles the angle is 2θ (**1 points**).

2. (7 points) Compute the hyperbolic area of the triangle with vertices $-\sqrt{3}+i$, $\sqrt{3}+i$, and ∞.

Solution: The triangle is the region enclosed by the circle of radius 2, and the lines with real part $\pm\sqrt{3}$ (**1 point**). The angle at ∞ is always 0 (**1 point**). Since this is symmetric about the imaginary axis, the angles at the finite vertices are equal, so it suffices to find the angle at $\sqrt{3}+i$. The radius at this point is given by the line $y = \frac{1}{\sqrt{3}}x$ and the tangent line is perpendicular to this line. Therefore the tangent line has slope $-\sqrt{3}$ and the intercept with the real axis is at $\sqrt{3}+1/\sqrt{3}$. In particular, by vertical angles, the angle β at $\sqrt{3}+i$ has tangent $1/\sqrt{3}$ and is thus $\pi/6$ (**2 points**).

Therefore, by Gauss-Bonnet, the area is $\pi - 2(\pi/6) = 2\pi/3$ (**3 points**). Alternately, one could award 3 points for acting by $\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$ to go from the circle of radius 2 to the unit circle and then 5 points for mimicing the proof from class - 2 for realizing that the limits of integration are $\pi/6 \leq x \leq 5\pi/6$ and 3 for computing the rest correctly. Hopefully you resisted the temptation to make everything look like the picture above - this would require looking at $\theta = \pi/12$, which is not one of the normally used trigonometric values.

3. (8 points) Give a continuous, surjective group homomorphism $f : \mathbf{R} \to \mathrm{SO}_2(\mathbf{R})$ with kernel $2\pi\mathbf{Z}$. Conclude that f induces an isomorphism of topological groups (in particular, a homeomorphism and a group isomorphism) between $\mathbf{R}/2\pi\mathbf{Z}$ and $\mathrm{SO}_2(\mathbf{R})$. Hint: If you do not use the angle addition formulas at any point, you're probably doing this exercise incorrectly.

Solution: Clearly the map $\theta \mapsto k_\theta$ is the one we seek. To say that this is a homomorphism would be to say that if $\alpha, \beta \in \mathbf{R}$ then $k_\alpha k_\beta = k_{\alpha+\beta}$. Therefore we compute $k_\alpha k_\beta$ to be

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{pmatrix}.$$

The angle addition formulas tell us that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ so we see that this is a homomorphism (**2 points**). Moreover, since both the sine and cosine functions are continuous on $\mathbf{R}$, we have a continuous map $f : \mathbf{R} \to M_2(\mathbf{R}) \cong \mathbf{R}^4$ landing in $\mathrm{SO}_2(\mathbf{R})$ (**2 points**). Now we just need to verify that the kernel is exactly $2\pi\mathbf{Z}$. Of course the trig functions are 2π-periodic so for all $n \in \mathbf{Z}$, $k_{2\pi n} = k_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore the kernel contains $2\pi\mathbf{Z}$. To show there is nothing else in the kernel we need to show that if $\theta \notin 2\pi\mathbf{Z}$ that k_θ is not the identity. For that, it suffices to show that $\cos(\theta) = 1$ only if $\theta \in 2\pi\mathbf{Z}$, and this is obvious just from looking at the graph of $\cos : \mathbf{R} \to \mathbf{R}$ (**2 points**).

Therefore, consider $\bar{f} : \mathbf{R}/2\pi\mathbf{Z} \to \mathrm{SO}_2(\mathbf{R})$ where $\bar{f}(\theta + 2\pi\mathbf{Z}) = f(\theta) = k_\theta$. Since f is continuous and surjective, $\bar{f}$ is also continuous and surjective, but it is also injective. Therefore, there is a continuous inverse homomorphism $k_\theta \mapsto \theta + 2\pi\mathbf{Z}$ and $\bar{f}$ is an isomorphism of topological groups (**2 points**). We can even explicitly see this (not a part of the question, but I think it's useful anyway!): if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SO}_2(\mathbf{R})$ then $a^2 + c^2 = 1$ so the point $a + ic$ lies on the unit circle. The Euclidean distance along the unit circle from $(1,0)$ to (a,c) is a continuous function of a and c (even differentiable - it's an arc length integral). This is our θ. Alternately we can explicitly see this as values of $\arccos(a) = \arcsin(c)$ on different domains.