Hyperbolic Geometry Homework 1

Name: Answer Rubric

1. (4 points) Let $\theta(t) = (t+1)\pi/3$ and $\gamma(t) = \cos(\theta(t)) + i\sin(\theta(t))$ for $0 \le t \le 1$. Compute the hyperbolic length
$$\int_\gamma \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{y} = \int_0^1 \frac{\sqrt{\mathrm{d}(\cos(\theta(t)))^2 + \mathrm{d}(\sin(\theta(t)))^2}}{\sin(\theta(t))}$$
by direct computation. Give both an <u>exact answer</u> and an approximation sufficient to distinguish from the answer to question 2. **Do not be scared to ask for help in office hours on this one. It's trickier than it looks.**

> **Solution:** First we use u-substitution with $u(t) = \theta(t)$ to find it is $\int_{\pi/3}^{2\pi/3} \frac{\mathrm{d}\theta}{\sin(\theta)}$ (**1 point**). An antiderivative for $\frac{1}{\sin\theta}$ is $\ln(\tan(\theta/2))$ (**1 point**). The tangent half-angle formula is $\tan(\theta/2) = \frac{\sin(\theta)}{1+\cos(\theta)}$ (**1 point**). Therefore the integral evaluates to (**1 point**)
> $$\ln\left(\frac{\sqrt{3}/2}{1+(-1/2)}\right) - \ln\left(\frac{\sqrt{3}/2}{1+(1/2)}\right) = \ln(\sqrt{3}) - \ln(1/\sqrt{3}) = \ln((\sqrt{3})^2) = \ln(3) \approx 1.1.$$

2. (2 points) Compare this to the hyperbolic length of $\gamma_0(t) = i\sqrt{3}/2 + (t-(1/2))$. Is $\gamma_0(t)$ a geodesic?

> **Solution:** Since $x = t - 1/2$ and $y = \sqrt{3}/2$, the length is $\int_0^1 \frac{\sqrt{(\mathrm{d}(\sqrt{3}/2))^2 + (\mathrm{d}(t-1/2))^2}}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}\int_0^1 \mathrm{d}t = 2/\sqrt{3} \approx 1.15$ (**1 point**). So of course γ_0 is not a geodesic(**1 point**).

3. (3 points) Verify that after applying the matrix $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ in $\mathrm{SO}_2(\mathbf{R})$, complex numbers of the form $\cos\theta + i\sin\theta$ (with $0 < \theta < \pi$) are moved to the imaginary axis. Use this to give the hyperbolic length of γ.

> **Solution:** (**2 points**)The matrix sends complex numbers of the form $\cos\theta + i\sin\theta$ to
> $$\begin{aligned} \frac{\frac{\cos\theta + i\sin\theta}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{\frac{\cos\theta + i\sin\theta}{\sqrt{2}} + \frac{1}{\sqrt{2}}} &= \frac{(\cos\theta - 1) + i\sin\theta}{(\cos\theta+1) + i\sin\theta} \\ &= \frac{(\cos\theta+1)(\cos\theta-1) + i[\sin\theta(\cos\theta+1) - \sin\theta(\cos\theta-1)] + \sin^2\theta}{(\cos\theta+1)^2 + \sin^2\theta} \\ &= \frac{\sin^2\theta + \cos^2\theta - 1 + i[2\sin\theta]}{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta} \\ &= i\frac{\sin\theta}{1+\cos\theta} = i\tan(\theta/2). \end{aligned}$$
> (**1 point**) Our arc goes from $\theta = \pi/3$ to $2\pi/3$. Since this transformation is an isometry, the length of γ is $\int_{\tan(\pi/6)}^{\tan(\pi/3)} \frac{\mathrm{d}y}{y} = \ln(\sqrt{3}) - \ln(1/\sqrt{3}) = \ln(3)$.

4. (5 points) Verify that if $a^2+b^2 \neq 1$ and $\theta = \frac{-1}{2}\arctan(2a/(a^2+b^2-1))$ then $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ moves $a+bi \in \mathcal{H}$ to the imaginary axis.

Solution: (**1 point**)Of course $a+bi$ gets moved to

$$\frac{(a+bi)\cos\theta - \sin\theta}{(a+bi)\sin\theta - \cos\theta} = \frac{(a\cos\theta - \sin\theta) + ib\cos\theta}{(a\sin\theta + \cos\theta) + ib\sin\theta}.$$

(**1 point**)We verify that the denominator is nonzero since $b > 0$ and if $\sin\theta = 0$ then $\cos\theta = \pm 1$, but more importantly $\sin(2\theta) = \tan(2\theta) = 0 = -2a/(a^2+b^2-1)$. Thus $a = 0$ if $\theta = 0$ so we were on the imaginary axis anyway.

(**1 point**)Assume now $a \neq 0$ so θ is not 0 or $\pm\pi/2$. We just need to verify that we have the correct real part, whose numerator is

$$(a\cos\theta - \sin\theta)(a\sin\theta + \cos\theta) + b^2\sin\theta\cos\theta = 0.$$

(**1 point**)This reduces to $(a^2+b^2-1)\cos\theta\sin\theta + a(\cos^2\theta - \sin^2\theta) = \frac{a^2+b^2-1}{2}\sin(2\theta) + a\cos(2\theta) = 0$. (**1 point**)This is true if and only if $\tan(2\theta) = \frac{-2a}{a^2+b^2-1}$. But this is essentially the definition of θ.

5. (3 points) Compute the length of γ one more time, this time with the cross-ratio. What method is easiest for you?

Solution: First we compute (**2 points**)

$$\begin{aligned} \lambda(-1, 1, \frac{-1+i\sqrt{3}}{2}, \frac{1+i\sqrt{3}}{2}) &= \frac{\frac{1+i\sqrt{3}}{2} - (-1)}{\frac{1+i\sqrt{3}}{2} - (1)} \frac{\frac{-1+i\sqrt{3}}{2} - (1)}{\frac{-1+i\sqrt{3}}{2} - (-1)} \\ &= \frac{(3+i\sqrt{3})(-3+i\sqrt{3})}{(-1+i\sqrt{3})(1+i\sqrt{3})} \\ &= \frac{-9-3}{-1-3} = 3. \end{aligned}$$

It follows that the length of γ is $\ln(\lambda) = \ln(3)$ (**1 point**).

6. (5 points) Verify that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R})$ with $cd \neq 0$ sends the imaginary axis to the open half-circle centered on the real axis with limit points a/c and b/d. Find the center and radius.

Solution: Assuming the imaginary axis is indeed sent to that circle, the diameter is the distance between a/c and b/d. Conveniently, $ad - bc = 1$ so $\mid (a/c) - (b/d) \mid = 1/ \mid cd \mid$. Therefore the circle we seek has center $b/d + 1/2cd = a/c - 1/2cd$ and radius $1/ \mid 2cd \mid$ (**2 points**). Therefore we want to verify that the point iy with $y > 0$ is sent to a point $u + iv$ where $v^2 + (u - a/c + 1/2cd)^2 = 1/(4c^2d^2)$. We know that $v = y/(c^2y^2+d^2)$ from class (**1 point**) and we can compute that $b = (ad-1)/c$ so $u = a/c - d/(c(c^2y^2+d^2))$(**1 point**. We are therefore left to compute that

$$\frac{1}{4c^2d^2} - \frac{1}{c^2(c^2y^2+d^2)} + \frac{d^2}{c^2(c^2y^2+d^2)^2} + \frac{y^2}{(c^2y^2+d^2)^2} = \frac{1}{4c^2d^2},$$

and this is an easy verification(**1 point**). Please do take off points if someone just says that since we're acting by an isometry it has to be to a circle. In class we only proved that isometries take geodesics to geodesics, not necessarily circles!