Some recent progress on the Frobenius Problem

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The Classical Frobenius Problem

Given positive coprime integers

$$a_1, \ldots, a_n$$

the *Frobenius Number*

$$g(a_1, \ldots, a_n)$$

is defined to be the largest integer M for which there are no non-negative integers

$$x_1, \ldots, x_n$$

such that

$$a_1x_1 + \cdots + a_nx_n = M.$$

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The *Frobenius Problem* is the problem of determining the Frobenius Number.

Classical work of Sylvester

If we think of the Frobenius Problem as asking about representing integers by the linear form

$$L(\vec{x}) = a_1x_1 + \cdots + a_nx_n$$

then when $n = 2$ exactly half the integers between 1 and $(a_1 - 1)(a_2 - 1)$ are representable by L.

Moreover, we have the following well-known identity :

$$g(a_1, a_2) = a_1a_2 - a_1 - a_2$$

We know a lot when $n = 2$

Beck and Robins were able to rederive the classical results of Sylvester as well as the following representability results when $n = 2$

- $g_k(a_1, a_2) = (k+1)a_1a_2 - a_1 - a_2$ where $g_k(a_1, a_2)$ denotes the largest k-representable integer
- If $k \geq 2$, the smallest k-representable integer(by L) is $a_1a_2(k-1)$
- If $k \geq 2$, the smallest interval containing all k-representable integers is $[g_{k-2}(a_1, a_2) + a_1 + a_2, g_k(a_1, a_2))]$
- Exactly $a_1a_2 - 1$ integers are uniquely representable
- For $k \geq 2$ exactly a_1a_2 integers are k-representable

Note that when $n = 2$ or $k = 0$ there's no difference between the largest integer representable *at most* k times and *exactly* k times.

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Not so much when $n \geq 3$

For $n \geq 3$ the Classical Frobenius problem becomes much less tractable, in fact NP Hard.

What little we know comes from either the asymptotic (this version due to Nathanson) on the number of representations of M by the linear form L

$$r_L(M) = \frac{M^{n-1}}{a_1 \dots a_n (n-1)!} + O(M^{n-2})$$

Or from the formula of Brauer and Shockley

$$g(a_1, da_2, \dots, da_n) = dg(a_1, a_2, \dots, a_n) + (d-1)a_1$$

Work with Brown et al.

In joint work with

- Alexander Brown
- Eleanor Dannenberg
- Jennifer Fox
- Joshua Hanna
- Katherine Keck
- Alexander Moore
- Zachary Robbins
- Brandon Samples

we found numerical evidence that a Brauer-Shockley type of theorem should hold for an appropriate generalization of the Frobenius Number.

One source of difficulties

If $n \geq 3$ we note that the asymptotic for $r_L(M)$ implies that there will be positive integers k where there is NO integer M which is representable in exactly k different ways.

Therefore there are two distinct generalizations of the quantity g_k (or if you prefer, k-representability):

- The largest integer which is representable in *at least k* different ways
- The largest integer which is representable in *exactly k* different ways if such an integer exists and it's either zero (or undefined) otherwise.

We used the second generalization because it made the following true:

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Our result

Theorem

If $k \geq 0$ either

$$g_k(a_1, da_2, \ldots, da_n) = d \cdot g_k(a_1, a_2, \ldots, a_n) + (d-1)a_1$$

or $g_k(a_1, da_2, \ldots, da_n) = g_j(a_1, a_2, \ldots, a_n) = 0$(or undefined).

We also discovered "discrepancies" or instances where $j < k$ but

$$0 < g_k(a_1, \ldots, a_n) < g_j(a_1, \ldots, a_n)$$

and these findings were published in the article “On a Generalization of the Frobenius Number” in January in the online Journal of Integer Sequences.

An Example

Let $a_1 = 3,\ a_2 = 5,\ a_3 = 8$

k	0	1	2	3	4	5	6	7	8	9
g_k	7	12	17	22	25	28	31	34	37	39

k	10	11	12	13	14	15	16	17	18	19
g_k	42	44	47	49	52	51	55	57	58	60

Besides being oddities, understanding the extent to which these discrepancies occur is key to understanding the interplay between the two generalizations of g_k and thus the general representability of positive integers by L.

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More on discrepancy

After publication, Jeffrey Shallit of the University of Waterloo discovered that the examples we produced had a peculiar property: that $a_1, \ldots, a_n$ were such that there was some i and some $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n \in \mathbf{Z}_{\geq 0}$ such that

$$a_i = a_1x_1 + \cdots + a_{i-1}x_{i-1} + a_{i+1}x_{i+1} + \cdots + a_nx_n$$

He called such tuples of coprime positive integers *unreasonable* since it was possible to use them to cook up trivial discrepancies such as:

$$g_0(4, 5, 8, 10) = 11$$

$$g_1(4, 5, 8, 10) = 9$$

Shallit produced many discrepancies, even for *reasonable* tuples of coprime positive integers, and offered to collaborate. The end result was the paper "Unbounded Discrepancy in Frobenius Numbers" to appear in INTEGERS. The following is the main theorem.

Theorem

1. *If* $n \geq 6$

$$g_0(2n-2, 2n-1, 2n, 3n-3, 3n) = n^2 - 3n + 1$$

2. *If* $k \geq 1,\ n > 6k+3,$

$$g_k(2n-2, 2n-1, 2n, 3n-3, 3n) = (6k+3)n - 1$$

A Quick idea of the proof of part 2

A key fact is that there are many possible "swaps" among a representation

$$a(2n-2)+b(2n-1)+c(2n)+d(3n-3)+e(3n)=M$$

e.g.

- $(a,b,c,d,e)\mapsto(a+3,b,c,d-2,e)$ or $(a,b,c,d,e)\mapsto(a,b,c+3,d,e-2)$
- $(a,b,c,d,e)\mapsto(a+1,b+1,c+1,d-1,e-1)$
- $(a,b,c,d,e)\mapsto(a,b+3,c,d-1,e-1)$
- $(a,b,c,d,e)\mapsto(a-1,b+2,c-1,d,e)$

So if M is k-representable, can show that $M\leq(2n-1)+2(2n)+(2k-1)3n=(6k+3)n-1$

Conclusions

So it's not merely that we can have reasonable tuples where $0 < g_1 < g_0$ or $0 < g_k < g_0$.

It's that the difference $g_0 - g_k$ can become arbitrarily large for any $k \geq 1$.

We also found a family in $n \geq 6$ variables where $g_0 - g_1$ can become unboundedly large and we can have $0 < g_1 < g_0$ in four variables.

It's still not known if we can have $0 < g_{k+1} < g_k$ for $k < 14$ in 3 variables.

There are some known examples where $g_2 < g_1 < g_0$

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