SOME THOUGHTS ON THE GROUP SCHEME PGL_n

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Recall that if R is a commutative ring with identity, $\mathrm{GL}_n(R) = \{(a_{ij}) : \det(a_{ij}) \neq 0\}$ is the group of matrices with R coefficients which are invertible. Proving this is an easy exercise using the classical adjoint of a matrix defined over any ring R, and in fact GL is a functor from the category of rings to the category of groups. Even better, it is a *sheaf* for the Zariski topology, as it is represented by the Hopf algebra $\mathbf{Z}[x_{ij}][1/\det]$.

What then can we say about $\mathrm{GL}_n(R)/Z(\mathrm{GL}_n(R)) = \mathrm{GL}_n(R)/R^\times$? Is it a functor from the category of rings to the category of groups? Is it a sheaf?

The answer to the first question is certainly yes. If $R \to S$ is a homomorphism of rings then we have a homomorphism $\mathrm{GL}_n(R) \to \mathrm{GL}_n(S)$ by the above (it amounts to applying the ring homomorphism to each coordinate). Moreover since $R \to S$ on units is $R^\times \to S^\times$ we have a map $\mathrm{GL}_n(R)/R^\times \to \mathrm{GL}_n(S)/S^\times$.

Thus is is at least a presheaf for the Zariski topology. It is however, not a sheaf as we show with this example with $\mathbf{Z}[\sqrt{-5}]$.

1. A concrete example

Consider the open cover $\mathbf{Z}[\sqrt{-5}][1/(6+5\sqrt{-5})]$, $\mathbf{Z}[\sqrt{-5}][1/(5+\sqrt{5})]$ of $\mathbf{Z}[\sqrt{-5}]$. Over these two rings respectively consider the matrices up to unit multiplication

$$M = \begin{pmatrix} \frac{2-3\sqrt{-5}}{6+5\sqrt{-5}} & \frac{5-4\sqrt{-5}}{6+5\sqrt{-5}} \\ \frac{5+3\sqrt{-5}}{6+5\sqrt{-5}} & 1 \end{pmatrix}, N = \begin{pmatrix} \frac{5-4\sqrt{-5}}{5+5\sqrt{-5}} & \frac{10-5\sqrt{-5}}{5+5\sqrt{-5}} \\ 1 & \frac{5+8\sqrt{-5}}{5+5\sqrt{-5}} \end{pmatrix}$$

We can compute that over $\mathbf{Q}(\sqrt{-5})$,

$$NM^{-1} = \begin{pmatrix} \frac{5-4\sqrt{-5}}{2-3\sqrt{-5}} & 0 \\ 0 & \frac{5-4\sqrt{-5}}{2-3\sqrt{-5}} \end{pmatrix}.$$

Over $\mathbf{Z}[\sqrt{-5}][1/(6+5\sqrt{-5})]$, $2-3\sqrt{-5}$ is invertible with inverse $\dfrac{-22-3\sqrt{-5}}{(6+5\sqrt{-5})^2}$. However, over that ring, $5-4\sqrt{-5}$ is not invertible and so these are distinct matrices even up to invertible scalar multiplication. It is however invertible over $R = \mathbf{Z}[\sqrt{-5}][1/(6+5\sqrt{-5}), 1/(5+5\sqrt{-5})]$, the scheme-theoretic intersection of the two covering open sets of $\mathbf{Z}[\sqrt{-5}]$. The inverse is $\dfrac{(5-4\sqrt{-5})(-10)(-22-3\sqrt{-5})}{(6+5\sqrt{-5})^2(5+5\sqrt{-5})^2}$ and therefore these two matrices give the same element of $\mathrm{GL}_n(R)/R^\times$. However, M and N cannot glue together to produce an element of $\mathrm{GL}_n(\mathbf{Z}[\sqrt{-5}])/(\pm 1)$, because there are too few units. Indeed we can produce representatives

$$\begin{pmatrix} 2-3\sqrt{-5} & 5-4\sqrt{-5} \\ 5+3\sqrt{-5} & 6+5\sqrt{-5} \end{pmatrix}, \begin{pmatrix} 5-4\sqrt{-5} & 10-5\sqrt{-5} \\ 5+5\sqrt{-5} & 5+8\sqrt{-5} \end{pmatrix}$$

for each over $\mathbf{Z}[\sqrt{-5}]$, but clearly these are not equal up to unit multiple. In fact, we shall shortly see that finding a matrix unique up to unit multiple for M and N would contradict the nontriviality of the ideal class group of $\mathbf{Z}[\sqrt{-5}]$. In any case the functor $R \to \mathrm{GL}_n(R)/R^\times$ is a presheaf that fails the gluing axiom of a sheaf.

2. As a sheaf

Consider $\mathbb{G}_m = \mathrm{GL}_n$ for $n = 1$. We showed above that the functor $\mathrm{GL}_n/\mathbb{G}_m$ is not a sheaf.

To fix this, we do what we always do when we have a presheaf which is not a sheaf: We sheafify! Rather, we find a sheafification, that is, a sheaf F such that $\mathrm{GL}_n/\mathbb{G}_m \to F$ is surjective. This does not mean simply that the map on global sections $\mathrm{GL}_n(R)/\mathbb{G}_m(R) \to F(R)$ is surjective, but rather that for all rings R there's an open cover $\{R_i\}$ of R such that $\mathrm{GL}_n(R_i)/\mathbb{G}_m(R_i) \to F(R_i)$ is surjective.

The short answer is that

$$F(-) = \mathrm{Hom}\left((\mathbf{Z}[x_{ij}][1/\det])_{\deg=0}, - \right).$$

The long answer is that we think of GL_n as $\mathbb{A}^{n^2} - \{\det = 0\}$ endowed with a group structure. When we try to define PGL_n we'd like to think of $\mathbb{P}^{n^2-1} - \{\det = 0\}$ endowed with that same group structure. We can do that however as the determinant polynomial

$$\det = \sum_{\tau \in S_n} \mathrm{sgn}(\tau) \prod_{i=1}^{n} x_{i,\tau(i)}$$

is homogeneous of degree n. Hence we have a natural grading on $\mathbf{Z}[x_{ij}][1/\det]$ by $\deg(f/\deg^e) = \deg(f) - en$ if f is homogeneous. Therefore the natural hope is that PGL_n is representable by $(\mathbf{Z}[x_{ij}][1/\det])_{\deg=0}$.

First let's show we have an injective map of sets $\mathrm{GL}_n(R)/\mathbb{G}_m(R) \to F(R)$, thus for each R we have $\mathrm{GL}_n(R)/\mathbb{G}_m(R)$ as a subgroup of $F(R)$. Let $(a_{ij}) \in \mathrm{GL}_n(R)$, or equivalently consider the homomorphism $\mathbf{Z}[x_{ij}][1/\det] \to R$ defined by sending x_{ij} to a_{ij}. By precomposing with the inclusion $\mathbf{Z}[x_{ij}][1/\det]_{\deg=0} \to \mathbf{Z}[x_{ij}][1/\det]$ is our map $\mathrm{GL}_n(R) \to F(R)$. This induces a map $\mathrm{GL}_n(R)/\mathbb{G}_m(R) \to F(R)$ because if f is homogeneous of degree en in $\mathbf{Z}[x_{ij}]$ then $f/\det^e$ is sent to $f(a_{ij})/\det(a_{ij})^e$ under the map given by $(a_{ij}) \in \mathrm{GL}_n(R)$ and thus if $\lambda \in \mathbb{G}_m(R) = R^\times$ then

$$\frac{f(\lambda a_{ij})}{\det(\lambda a_{ij})^e} = \frac{\lambda^{en} f(a_{ij})}{(\lambda^n \det(a_{ij}))^e} = \frac{f(a_{ij})}{\det(a_{ij})^e}.$$

To prove the map is injective, suppose (a_{ij}) and (b_{ij}) are in $\mathrm{GL}_n(R)$ mapping to the same $m \in F(R)$ and there is no $\lambda \in \mathbb{G}_m(R)$ such that $a_{ij} = \lambda b_{ij}$ for all i, j. Without loss of generality we may assume they have determinant 1 (by multiplying a column in each matrix by $1/\det$). Since $(a_{ij}) \in \mathrm{GL}_n(R)$, we can find i_0, j_0 such that a_{i_0,j_0} is not a zero-divisor(in fact we can find a $\sigma \in S_n$ such that $\prod_{i=1}^n x_{i\sigma(i)}$ is not a zero-divisor, else the determinant is a zero-divisor and thus not invertible). Therefore $m\left(\frac{x_{i_0j_0}^n}{\det}\right) = a_{i_0j_0}^n = b_{i_0j_0}^n$ and $a_{i_0j_0} = \zeta b_{i_0j_0}$ for some $\zeta \in \mu_n(R[1/b_{i_0j_0}]) = \mu_n(R)$. Likewise evaluating m at $\frac{x_{ij}^n}{\det}$ and $\frac{x_{ij}x_{i_0j_0}^{n-1}}{\det}$ shows that

$$a_{ij} b_{i_0j_0}^n = a_{ij} a_{i_0j_0}^n = b_{ij} b_{i_0j_0}^{n-1} a_{i_0j_0}$$

and thus $a_{ij} = b_{ij}\zeta$ for all i, j. Note that along the way, we've also proven that $\mathrm{SL}_n(R)/\mu_n(R)$ injects into $F(R)$.

Now that we've shown we have an injective map of sets, let's prove surjectivity as a sheaf. To do this, let's recall that if we have a homomorphism $m : \mathbf{Z}[x_{ij}][1/\det] \to R$ then we have a very natural open cover of R. Namely if $\sigma \in S_n$, define

$$t_\sigma = \frac{\prod_{k=1}^n x_{k\sigma(k)}}{\det}, r_\sigma = m(t_\sigma).$$

Quite naturally then

$$1 = m\left(\sum_\sigma \mathrm{sgn}(\sigma) t_\sigma\right) = \sum_\sigma \mathrm{sgn}(\sigma) r_\sigma,$$

so $R[1/r_\sigma]$ forms a natural open cover of R.

Now consider our situation summed up in the following diagram:

$$\begin{array}{ccc} \mathbf{Z}[x_{ij}][1/\det]_0 & \hookrightarrow & \mathbf{Z}[x_{ij}][1/\det] \\ m\downarrow & & ? \\ R & \to & R[1/r_\sigma] \end{array}$$

We will actually supply n different homomorphisms $\mathbf{Z}[x_{ij}][1/\det] \to R[1/r_\sigma]$ making the above diagram commute. The basic idea will be to fix $1 \le k \le n$ and send $x_{k,\sigma(k)}$ to 1[1] and thereby determine the entire homomorphism as follows:

First, recall that

$$\frac{x_{ij}}{x_{k\sigma(k)}} = \frac{x_{ij}\prod_{\ell\neq k} x_{\ell\sigma(\ell)}}{\prod_{\ell=1}^n x_{\ell\sigma(\ell)}} = \frac{\frac{x_{ij}\prod_{\ell\neq k} x_{\ell\sigma(\ell)}}{\det}}{t_\sigma}$$

Therefore if we define a map of sets $m_{k,\sigma} : \mathbf{Z}[x_{ij}][1/\det] \to R[1/r_\sigma]$ by $x_{ij} \mapsto \dfrac{m\left(\frac{x_{ij}\prod_{\ell\neq k} x_{\ell\sigma(\ell)}}{\det}\right)}{r_\sigma}$ then we will show it commutes with $\mathbf{Z}[x_{ij}][1/\det]_0 \xrightarrow{m} R \to R[1/r_\sigma]$ and is thus a homomorphism, and a realization that the restriction of m to $R[1/r_\sigma]$ is an element of $\mathrm{GL}_n(R[1/r_\sigma])/\mathbb{G}_m(R[1/r_\sigma])$.

Second, given that definition of $m_{k,\sigma}$, it is natural to think of $m_{k,\sigma}(\det)$ as $\sum_{\tau\in S_n} \mathrm{sgn}(\tau)\prod_{i=1}^n m\left(\frac{x_{i,\tau(i)}}{x_{k,\sigma(k)}}\right) = m\left(\frac{\det}{x^n_{k,\sigma(k)}}\right)$ even though the latter is not defined. The point is that $m\left(\frac{x^n_{k,\sigma(k)}}{\det}\right)$ is defined and so naturally

$$m_{k,\sigma}\left(\frac{1}{\det^e}\right) = \frac{m\left(\frac{x^{en}_{k,\sigma(k)}}{\det^e}\right)}{1}$$

for any $e \in \mathbf{Z}_{\geq 0}$.

Finally we prove that $m_{k,\sigma}$ makes the following diagram commute:

$$\begin{array}{ccc} \mathbf{Z}[x_{ij}][1/\det]_0 & \hookrightarrow & \mathbf{Z}[x_{ij}][1/\det] \\ m\downarrow & & m_{k,\sigma}\downarrow \\ R & \to & R[1/r_\sigma] \end{array}$$

[1]Technically here r_σ might be a zero divisor, but considering $R[1/r_\sigma]$ as an abstract localization we'll send $x_{k,\sigma(k)}$ to r_σ/r_σ. We don't seriously consider the case that r_σ is a zero divisor as we could in that case just exclude it and get a smaller open cover, but we proceed as we do only to give a unified treatment.

Let $f = \sum_{\mathbf{s}} a_{\mathbf{s}} \prod x_{ij}^{s_{ij}}$ be homogeneous of degree en with integer coefficients. Clearly under $\mathbf{Z}[x_{ij}][1/\det]_0 \to R \to R[1/r_\sigma]$, $\frac{f}{\det^e}$ is mapped to

$$\frac{\left(\sum_{\mathbf{s}} a_{\mathbf{s}} m\left(\frac{\prod x_{ij}^{s_{ij}}}{\det^e}\right)\right)}{1}.$$

Now including $\frac{f}{\det^e}$ into $\mathbf{Z}[x_{ij}][1/\det]$, we observe that

$$\begin{aligned}
m_{k,\sigma}\left(\frac{f}{\det^e}\right) &= \left(m_{k,\sigma(k)}\left(\frac{1}{\det^e}\right)\right)\sum_{\mathbf{s}} a_{\mathbf{s}} \prod m_{k,\sigma}\left(x_{ij}^{s_{ij}}\right) \\
&= \left(m_{k,\sigma(k)}\left(\frac{1}{\det^e}\right)\right)\sum_{\mathbf{s}} a_{\mathbf{s}} \prod \frac{m\left(\frac{x_{ij}\prod_{\ell\neq k} x_{\ell\sigma(\ell)}}{\det}\right)^{s_{ij}}}{r_\sigma^{s_{ij}}} \\
&= \frac{m\left(\frac{x_{k,\sigma(k)}^{en}}{\det^e}\right)}{r_\sigma^{en}}\sum_{\mathbf{s}} a_{\mathbf{s}} \prod m\left(\frac{x_{ij}\prod_{\ell\neq k} x_{\ell\sigma(\ell)}}{\det}\right)^{s_{ij}} \\
&= \frac{\sum_{\mathbf{s}} a_{\mathbf{s}} m\left(\frac{(\prod x_{ij}^{s_{ij}})\prod_{\ell=1}^{n} x_{\ell,\sigma(\ell)}^{en}}{\det^{e(n+1)}}\right)}{r_\sigma^{en}} \\
&= \frac{m\left(\frac{\prod_{\ell=1}^{n} x_{\ell,\sigma(\ell)}^{en}}{\det^{en}}\right) m\left(\sum_{\mathbf{s}} a_{\mathbf{s}}\frac{\prod x_{ij}^{s_{ij}}}{\det^e}\right)}{r_\sigma^{en}} \\
&= \frac{r_\sigma^{en}\sum_{\mathbf{s}} a_{\mathbf{s}} m\left(\frac{\prod x_{ij}^{s_{ij}}}{\det^e}\right)}{r_\sigma^{en}} \\
&= \frac{\sum_{\mathbf{s}} a_{\mathbf{s}} m\left(\frac{\prod x_{ij}^{s_{ij}}}{\det^e}\right)}{1}
\end{aligned}$$

Thus $m_{k,\sigma}$ commutes with m, which shows that there is an open cover of R on which every restriction of m lands in $\mathrm{GL}_n/\mathbb{G}_m$, which proves that GL_n surjects onto our functor F, which we may now feel empowered to call PGL_n.

3. Revisiting a concrete example

We have thus far paid a lot of lip service to the idea that the natural inclusion $\mathrm{GL}_n(R)/\mathbb{G}_m(R) \hookrightarrow \mathrm{PGL}_n(R)$ might not be surjective for some rings R. The fact is, most of the rings to which we are first introduced belong to a wide class of rings for which this inclusion must also be surjective.

Define the Picard Group $\mathrm{Pic}(R)$ of a ring R to be the set of rank one projective R-modules up to isomorphism, made into a group with identity R, multiplication provided by the tensor product $\otimes_R$ and inversion provided by taking the dual R-module. Another way to characterize this group is as parameterizing line bundles over the scheme $\mathrm{Spec}(R)$. From this perspective, we see that $\mathrm{Pic}(R) = \check{\mathrm{H}}^1(\mathrm{Spec}(R), \mathbb{G}_m \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(R))$, where $\check{\mathrm{H}}^1(X, F)$ denotes the full first Čech cohomology group of the sheaf F on X, that is, the limit over all covers $\mathfrak{U}$ of X of the Čech cohomology of F over $\mathfrak{U}$. If we recall that sheaf cohomology produces the same cohomology groups as Čech cohomology and consider $\mathbb{G}_m$, GL_n and PGL_n

to denote $\mathbb{G}_m \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(R)$, $\mathrm{GL}_n \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(R)$ and $\mathrm{PGL}_n \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(R)$ as needed we have the following:

Theorem 1. *If* $\mathrm{Pic}(R) = 0$ *then* $\mathrm{PGL}_n(R) = \mathrm{GL}_n(R)/\mathbb{G}_m(R) = \mathrm{GL}_n(R)/R^\times$.

The proof is simple. We already established that there is a surjective map of sheaves $\mathrm{GL}_n \to \mathrm{PGL}_n$ such that for all R the kernel of $\mathrm{GL}_n(R) \to \mathrm{PGL}_n(R)$ is $\mathbb{G}_m(R)$, therefore we have the following exact sequence of sheaves

$$0 \to \mathbb{G}_m \to \mathrm{GL}_n \to PGL \to 0.$$

Applying the functor Γ of global sections gives us an exact sequence of groups

$$0 \to \mathbb{G}_m(R) \to \mathrm{GL}_n(R) \to \mathrm{PGL}_n(R) \to \mathrm{H}^1(\mathrm{Spec}(R), \mathbb{G}_m) \to \dots$$

However if $\mathrm{Pic}(R) = 0$ then we have $\mathrm{H}^1(\mathrm{Spec}(R), \mathbb{G}_m) = \check{\mathrm{H}}^1(\mathrm{Spec}(R), \mathbb{G}_m) = 0$, hence

$$0 \to \mathbb{G}_m(R) \to \mathrm{GL}_n(R) \to \mathrm{PGL}_n(R) \to 0$$

is exact.

This shows that if there is element of $\mathrm{PGL}_n(R)$ which is not an element of $\mathrm{GL}_n(R)/\mathbb{G}_m(R)$ then $\mathrm{Pic}(R)$ must be nonzero. To produce one of these we recall that if R is a Dedekind domain, $\mathrm{Pic}(R)$ is the ideal class group of R, which brings us back to the concrete example. The ring $\mathbf{Z}[\sqrt{-5}]$ is the most commonly known ring with a nontrivial Picard group, in fact $\mathrm{Pic}(\mathbf{Z}[\sqrt{-5}]) \cong \mathbf{Z}/2\mathbf{Z}$.

Defining a homomorphism $\mathbf{Z}[a,b,c,d][1/\det]_0 \to \mathbf{Z}[\sqrt{-5}]$ is equivalent to giving a compatible system of assignments of the degree 2 monomials to elements of $\mathbf{Z}[\sqrt{-5}]$ such that $ad - bc \in \mathbf{Z}[\sqrt{-5}]^\times = \{\pm 1\}$. A natural way to make such an assignment which does not give an element of $\mathrm{GL}_n(\mathbf{Z}[\sqrt{-5}])/\{\pm 1\}$ would be to think of assigning each of a, b,c and d to nonprincipal fractional ideals. In essence we are building a "pseudomatrix"

$$\begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

where the entries are ideals. Since $\mathrm{Pic}(\mathbf{Z}[\sqrt{-5}]) \cong \mathbf{Z}/2\mathbf{Z}$, each will be in the same 2-torsion ideal class and so assuming we pick ideals such that the generator of $I_{11}I_{22}$ minus the generator of $I_{12}I_{21}$ is 1 without loss of generality we should be able to generate a "non-principal" element of $\mathrm{PGL}_n(\mathbf{Z}[\sqrt{-5}])$.

In this light consider the pseudomatrix

$$\begin{pmatrix} (7, 6+5\sqrt{-5}) & (\sqrt{-5})(3, 1+\sqrt{-5}) \\ (\sqrt{-5})(2, 1+\sqrt{-5}) & (23, 6+5\sqrt{-5}) \end{pmatrix}$$

and note that this would encourage an assignment $ad \mapsto 6 + 5\sqrt{-5}$, $bc \mapsto 5 + 5\sqrt{-5}$, so that $ad - bc \mapsto 1$. After making a few choices of which principal generator to use (for instance, $c^2 \mapsto -10$ seemed like a natural choice) everything is determined and a nontrivial amount of calculation in relatively trivial algebraic number theory yields the following:

ad	$6+5\sqrt{-5}$
bc	$5+5\sqrt{-5}$
a^2	$2-3\sqrt{-5}$
b^2	$10-5\sqrt{-5}$
c^2	-10
d^2	$-22-3\sqrt{-5}$
ab	$5-4\sqrt{-5}$
ac	$5+3\sqrt{-5}$
bd	$5+8\sqrt{-5}$
cd	$-15-\sqrt{-5}$

Note that working with ideal classes ensures that these satisfy all the degree 4 relations between these degree 2 monomials (e.g. $(ab)^2 = a^2b^2, (ad)(bc) = (ab)(cd) = (ac)(bd)$, etc.) and so we have a well defined homomorphism which cannot lift to an element of $\mathrm{GL}_n(\mathbf{Z}[\sqrt{-5}])/\pm 1$ as we cannot find principal generators for any of the nonprincipal ideals in the pseudomatrix.

In general if R is a Dedekind domain and $I_{11}, I_{12}, \ldots, I_{nn}$ are non principal ideals such that every n-fold product of them is principal and there are principal generators g_σ of the ideals $\prod_{i=1}^n I_{i\sigma(i)}$ are such that $\sum_\sigma \mathrm{sgn}(\sigma)g_\sigma$ is invertible, then it appears that we can produce an element of $\mathrm{PGL}_n(R)$ which does not lie in $\mathrm{GL}_n(R)/R^\times$. Our method will essentially be copying the work in the case $n = 2$ by sending each degree n monomial $\prod x_{ij}^{s_{ij}}$ to a principal generator of $\prod I_{ij}^{s_{ij}}$ in a compatible way. Such a generator is unique up to multiplication by $R^\times$ and the only lingering question is what exactly is forced on us by "compatibility".