# THE RIEMANN-ROCH THEOREM

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## 1. Introduction

The Riemann-Roch theorem is a classical result relating the zeros and poles of a function on a curve. In particular, given constraints limiting where poles and zeros can be and of what order, the Riemann-Roch theorem provides the dimension of the space of functions satisfying that condition. For instance, it tells us that on the Riemann Sphere, the rational functions in $\mathbb{C}(z)$ which have a pole of order at most one at 0 and no other poles are the functions $a + b/z$ for $a, b \in \mathbb{C}$. In particular, the dimension of this space over $\mathbb{C}$ is two.

Originally developed in complex function theory, Bernhard Riemann and his student Gustav Roch proved the theorem for Riemann surfaces in a purely analytic context. The legitimacy of the proof was called into question by Weierstrass, who found a counterexample to a main tool in the proof that Riemann called Dirichlet's principle. In spite of this challenge, the theorem's ideas were too useful to do without. In 1882, Dedekind and Weber gave a wholly new proof on an ideal-theoretic foundation. Since that time, Hilbert rigorously stated and proved Dirichlet's principle for a specific class of functions, justifying the whole of Riemann's proof. In this paper we will follow the methods of Andre Weil to obtain a proof of the Riemann-Roch theorem for curves over a general algebraically closed field.

In studying the geometry of a curve, the field of rational functions on that curve can be defined and by studying that field we may prove the Riemann-Roch theorem. In this paper we will forego the geometry of a curve and directly study the fields we may encounter as fields of rational functions.

## 2. Function Fields and Valuations

**Definition 2.1.** For a field $F$, a *function field* over $F$ (in one variable) is a field $K$ of transcendence degree one over $F$. We further require that $F$ is algebraically closed in $K$ and $K$ is finitely generated over $F$.

**Example 2.2.** The rational function field $F(u)$ with $u$ transcendental over $F$ is a function field over $F$. We have $F$ algebraically closed in $F(u)$, because if we let $E$ be the algebraic closure of $F$ in $F(u)$, then $E(u) = F(u)$ and so $[E(u) : F(u)] = 1$. By Lemma 7.4.4 in [3], if $E$ is an extension of $F$ and $[E(u) : F(u)]$ is finite then $[E : F] = [E(u) : F(u)] = 1$.

**Example 2.3.** A nonexample of a function field is $\mathbb{C}(u)/\mathbb{R}$ because $\mathbb{R}$ is not algebraically closed in $\mathbb{C}(u)$.

For any transcendental $x \in K - F$, the transcendence degree

$$\operatorname{tr}\deg(K/F) = \operatorname{tr}\deg(K/F(x)) + \operatorname{tr}\deg(F(x)/F).$$

Since $\operatorname{tr}\deg(K/F) = \operatorname{tr}\deg(F(x)/F) = 1$ we must have $\operatorname{tr}\deg(K/F(x)) = 0$. Thus $K$ is algebraic over $F(x)$ and since $K$ is finitely generated, we have $[K : F(x)] < \infty$ for any $x \in K - F$.

**Definition 2.4.** By a (*discrete*) *valuation* on a field $K$ we will mean a surjective function $v : K^\times \twoheadrightarrow \mathbb{Z}$ satisfying

- $v(xy) = v(x) + v(y)$,
- $v(x + y) \geq \min\{v(x), v(y)\}$.

We extend $v$ to 0 by setting $v(0) = \infty$, so $v(0) > v(x)$ for any $x \in K^\times$.

**Example 2.5.** The ring of integers $\mathbb{Z}$ is a unique factorization domain, so any nonzero element $n$ may be written be written as a product of primes $\pm p_1^{e_1} \dots p_g^{e_g}$. For any specific prime $p$, define the valuation $\operatorname{ord}_p$ by

$$\operatorname{ord}_p(n) = \begin{cases} e_i & \text{if } p = p_i \\ 0 & \text{if } p \neq p_1, \dots, p_g. \end{cases}$$

We may extend any such function to $\mathbb{Q}^\times$ by setting $\operatorname{ord}_p(\frac{n}{m}) = \operatorname{ord}_p(n) - \operatorname{ord}_p(m)$ which is well-defined because if $\frac{l}{k} = \frac{n}{m}$ for any nonzero $k, l \in \mathbb{Z}$, then $nk = lm$. Then

$$\operatorname{ord}_p(n) + \operatorname{ord}_p(k) = \operatorname{ord}_p(l) + \operatorname{ord}_p(m),$$

and so

$$\operatorname{ord}_p\left(\frac{l}{k}\right) = \operatorname{ord}_p(l) - \operatorname{ord}_p(k) = \operatorname{ord}_p(n) - \operatorname{ord}_p(m) = \operatorname{ord}_p\left(\frac{n}{m}\right).$$

**Example 2.6.** We know that since $\mathbb{C}$ is algebraically closed, any nonzero polynomial in $\mathbb{C}[x]$ decomposes into a finite product of linear parts as $f(x) = c(x - a_1)^{e_1} \dots (x - a_g)^{e_g}$. Linear polynomials are exactly the irreducible or prime elements of $\mathbb{C}[x]$. As such, for any $a \in \mathbb{C}$ we can define the function $\operatorname{ord}_{x-a}$ by

$$\operatorname{ord}_{x-a}(f(x)) = \begin{cases} e_i & \text{if } a = a_i \\ 0 & \text{if } a \neq a_1, \dots, a_g. \end{cases}$$

Using the same methods as is Example 2.5, we can extend these functions to $\mathbb{C}(x)^\times$, yielding discrete valuations on $\mathbb{C}(x)$. Another discrete valuation on $\mathbb{C}(x)$ is the negative degree valuation $\operatorname{ord}_\infty(f(x)) = -\deg(f(x))$. The notation $\operatorname{ord}_\infty$ is used because of its role in complex analysis. Furthermore, adopting the conventions of complex analysis, we will write $\operatorname{ord}_{x-a}$ as $\operatorname{ord}_a$, because it measures the order of vanishing of a rational function at $a$.

To any discrete valuation $v$ on $K$ we have a discrete valuation ring

$$\mathrm{O}_v := \{x \in K : v(x) \geq 0\}$$

with unique maximal ideal

$$\mathfrak{m}_v := \{x \in K : v(x) \geq 1\}.$$

**Lemma 2.7.** *The valuation ideal $\mathfrak{m}_v$ is principal.*

*Proof.* Let $t$ be an element of $\mathrm{O}_v$ such that $v(t) = 1$. Clearly $t\,\mathrm{O}_v \subset \mathfrak{m}_v$ because if $f \in \mathrm{O}_v$ then $v(tf) = 1 + v(f) \geq 0$. If we consider any $x \in \mathfrak{m}_v$, then $v(x) = k \geq 1$ and $v(x/t) = k - 1 \geq 1 - 1 = 0$ and so $x/t \in \mathrm{O}_v$. Therefore $x = t(x/t) \in t\,\mathrm{O}_v$.

□

**Definition 2.8.** If $t$ is a generator of $\mathfrak{m}_v$ we will call it a *uniformizing parameter.*

**Definition 2.9.** A *place* of a function field $K$ over an algebraically closed field $F$ is a discrete valuation $v$ on $K$ which is trivial on $F$(i.e. $v(F^\times) = 0$. Because of examples from complex analysis where valuations correspond to points, we usually write a valuation on $K$ trivial on $F$ as $P$, with the valuation of $f \in K$ at $P$ written as $\mathrm{ord}_P(f)$.

In this light, if $\mathrm{ord}_P(f) > 0$ then we say that $f$ has a *zero* of order $\mathrm{ord}_P(f)$ at $P$ and if $\mathrm{ord}_P(f) < 0$ then $f$ has a *pole* of order $-\mathrm{ord}_P(f)$ at $P$. More generally we call $|\mathrm{ord}_P(f)|$ the *multiplicity* of $f$ at $P$.

**Example 2.10.** The valuations listed in Example 2.6 are all trivial on $\mathbb{C}$ and are thus places of $\mathbb{C}(x)$. It was also commented that for $a \in \mathbb{C}$, $\mathrm{ord}_a(f)$ measures the order of vanishing of $f$ at $a$. Let us now examine why we give the name $\mathrm{ord}_\infty$ to $-\deg$.

For any $r(x) = \frac{f(x)}{g(x)} \in \mathbb{C}(x)$ with $f(x), g(x) \in \mathbb{C}[x]$, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ if and only if $\deg(f(x)) < \deg(g(x)$, i.e. if $-\deg(f(x)) + \deg(g(x)) = \mathrm{ord}_\infty(\frac{f(x)}{g(x)}) > 0$. We can then see via limits that $\mathrm{ord}_\infty$ measures the order of vanishing of $r(x)$ at the point $\infty$ on the Riemann Sphere.

**Example 2.11.** Let $P$ be a place on $K = F(x)$ with $F$ algebraically closed. We will show that $P = \mathrm{ord}_a$ for a unique $a \in F$ or $-\deg = \mathrm{ord}_\infty$. Since $P$ is a place, $\mathrm{ord}_P$ is not identically zero on $K^\times$ and so we can find some irreducible $\pi(x)$ in $K^\times$ with $\mathrm{ord}_P(\pi(x)) \neq 0$.

If $\mathrm{ord}_P(x) \geq 0$ we will show that $\mathrm{ord}_P = \mathrm{ord}_a$ (as in Example 2.6 where $F = \mathbb{C}$) for a unique $a \in F$. Since $F$ is algebraically closed, $\pi(x) = x - a$ for some $a \in F$. Note that according the properties of a valuation, $\mathrm{ord}_P$ takes non-negative values on $F[x]$ if $\mathrm{ord}_P(x) \geq 0$. In particular $\mathrm{ord}_P(x - a) \geq 0$ and because $\mathrm{ord}_P(x - a) \neq 0$, $\mathrm{ord}_P(x - a) \geq 1$. We claim that $a$ is unique. Suppose to the contrary that there were an element $b \in F$ where $b \neq a$ and $\mathrm{ord}_P(x - b) \geq 1$. Then $\mathrm{ord}_P((x - a) - (x - b)) \geq 1$, but this is impossible because $(x - a) - (x - b) = b - a \in F^\times$ and $\mathrm{ord}_P$ is trivial on $F^\times$. So for any

$b \in F$ such that $b \neq a$, $\operatorname{ord}_P(x-b) = 0$. Now for any $f(x) \in F[x] - \{0\}$, we can factor $f(x) = (x-a)^n g(x)$ where $(x-a)$ does not divide $g(x)$. Then

$$\operatorname{ord}_P(f(x)) = n \operatorname{ord}_P(x-a) + \operatorname{ord}_P(g(x)) = n \operatorname{ord}_P(x-a).$$

We know $\operatorname{ord}_P(g(x)) = 0$ because $g(x)$ will decompose into a product of irreducibles $\prod_i (x-b_i)^{n_i}$ where $b_i \neq a$ for any $i$, and so $\operatorname{ord}_P(g(x)) = \sum_i n_i \operatorname{ord}_P(x-b_i) = \sum_i 0 = 0$ and so $\operatorname{ord}_P(f(x)) = \operatorname{ord}_a(f(x)) \operatorname{ord}_P(x-a)$ for $f(x) \in F[x] - \{0\}$. This formula is multiplicative, so we may find that this formula extends to all of $K^\times$, i.e.

$$\operatorname{ord}_P(K^\times) = \operatorname{ord}_P(x-a) \operatorname{ord}_a(K^\times) = \operatorname{ord}_P(x-a)\mathbb{Z}.$$

Since $\operatorname{ord}_P(K^\times) = \mathbb{Z}$, $\operatorname{ord}_P(x-a)$ is a unit in $\mathbb{Z}$ and since $\operatorname{ord}_P(x-a) \geq 0$, $\operatorname{ord}_P(x-a) = 1$. Thus $\operatorname{ord}_P = \operatorname{ord}_a$ and we have proved what we wished if $\operatorname{ord}_P(x) \geq 0$.

Now if $\operatorname{ord}_P(x) < 0$, we will show that $\operatorname{ord}_P = -\deg$. Since $\operatorname{ord}_P(x) < 0$, $\operatorname{ord}_P(1/x) > 0$. We can run through the same argument with $F[1/x]$ replacing $F[x]$, so that the only irreducible of $F[1/x]$ on which $\operatorname{ord}_P$ takes nonzero values is $1/x$ and $\operatorname{ord}_P(1/x) = 1$. We then see that on $F[1/x] - \{0\}$,

$$\operatorname{ord}_P(f(1/x)) = \operatorname{ord}_{1/x}(f(1/x)) \operatorname{ord}_P(1/x) = \operatorname{ord}_{1/x}(f(1/x)).$$

If we extend multiplicatively, we find that $\operatorname{ord}_P = \operatorname{ord}_{1/x}$ over all of $K^\times$ and our result follows from the fact that $\operatorname{ord}_P(x) = -1 = -\deg(x)$.

With this example in mind, we can come to understand the structure of function fields and their discrete valuations. In the number field case, we can think about the discrete valuations as coming from primes of the integral closure of $\mathbb{Z}$. In the function field case, we can draw many of the discrete valuations on $K$ from primes in a subring of $K$. We can see however that even in the case $K = F(x)$ we can't get the full picture from just one set of primes. We can however easily describe all of the places of $K$.

**Theorem 2.12.** *Let $K$ be a function field and let $x \in K - F$. The places of $K$ which are non-negative on $x$ are in bijection with the primes of $R_x$, the integral closure of $F[x]$ in $K$. In particular, for a place $P$ and its corresponding prime $\mathfrak{p}$, $(R_x)_\mathfrak{p} = \mathrm{O}_P$.*

*Proof.* If we are given a place $P$ for which $\operatorname{ord}_P(x) \geq 0$, let us look at $\mathfrak{m}_P \cap R_x$. Since $x \in \mathrm{O}_P$, $F[x] \subset \mathrm{O}_P$ and since $\mathrm{O}_P$ is a discrete valuation ring, it is integrally closed, so $R_x \subset \mathrm{O}_P$. We know that $\mathfrak{m}_P \cap R_x \neq \{0\}$ because if that were the case, $R_x - \{0\} \subset \mathrm{O}_P - \mathfrak{m}_P = \mathrm{O}_P^\times$. In that case ratios of elements in $R_x - \{0\}$ are in $\mathrm{O}_P^\times$, in particular $F(x)$ is a subset of $\mathrm{O}_P$ and so is its integral closure. The integral closure of $F(x)$ is however the algebraic closure of $F(x)$, i.e. $K$. This cannot happen because if $K \subset \mathrm{O}_P$ then $\operatorname{ord}_P$ takes only non-negative values and is therefore not onto $\mathbb{Z}$. Since $\mathfrak{m}_P \cap R_x$ is not zero, it is a prime ideal of $R_x$.

Now suppose we are given a prime ideal $\mathfrak{p}$ of $R_x$. Since $R_x$ is the integral closure of the PID $F[x]$ in a finite, algebraic extension of its fraction field, $R_x$

is a Dedekind Domain. Therefore if we pick any $y \in R_x - \{0\}$, we can factor $yR_x$ uniquely into a product of prime ideals $\prod_{\mathfrak{q}} \mathfrak{q}^{k_{\mathfrak{q}}}$. Define the function $\mathrm{ord}_{\mathfrak{q}}(y) := k_{\mathfrak{q}}$ for all primes $\mathfrak{q}$. Now take $z \in R_x - \{0\}$ so that $zR_x = \prod_{\mathfrak{q}} \mathfrak{q}^{l_{\mathfrak{q}}}$. Since

$$yzR_x = (yR_x)(zR_x) = \prod_{\mathfrak{q}} \mathfrak{q}^{k_{\mathfrak{q}}+l_{\mathfrak{q}}},$$

$$\mathrm{ord}_{\mathfrak{q}}(yz) = k_{\mathfrak{q}} + l_{\mathfrak{q}} = \mathrm{ord}_{\mathfrak{q}}(y) + \mathrm{ord}_{\mathfrak{q}}(z).$$

If we extend this function to $K^\times$ via $\mathrm{ord}_{\mathfrak{q}}(y/z) = \mathrm{ord}_{\mathfrak{q}}(y) - \mathrm{ord}_{\mathfrak{q}}(z)$ this property will hold because

$$1/zR_x = \prod_{\mathfrak{q}} \mathfrak{q}^{-l_{\mathfrak{q}}}$$

as a fractional ideal.

We then have a group homomorphism between $K^\times$ and $\mathbb{Z}$ for any given prime $\mathfrak{p}$. This homomorphism is onto because for each prime $\mathfrak{p}$ there is some element $t \in R_x$ with $\mathrm{ord}_{\mathfrak{p}}(t) = 1$. If there were no such element, then $\mathfrak{p} - \mathfrak{p}^2$ is empty and so $\mathfrak{p} = \mathfrak{p}^2$. By definition, $\mathfrak{p}^2$ is made up of finite sums of the product of two elements of $\mathfrak{p} = \mathfrak{p}^2$. Therefore $\mathfrak{p} = \mathfrak{p}^2 = \mathfrak{p}^4$. If we repeat this process, we find $\mathfrak{p} = \cap_{n>0}\mathfrak{p}^n$. Since $R_x$ is Dedekind, it is Noetherian and since $\mathfrak{p}$ is a proper ideal of $R_x$, $\mathfrak{p} = \cap_{n>0}\mathfrak{p}^n = (0)$ and is therefore not prime by the Krull intersection theorem.

Now let $y, z \in K^\times$ and $m = \min\{\mathrm{ord}_{\mathfrak{p}}(y), \mathrm{ord}_{\mathfrak{p}}(z)\}$. If $m \geq 0$ then $y, z \in \mathfrak{p}^m \subset R_x$. Since $\mathfrak{p}^m$ is an ideal it is closed under addition, so $y+z \in \mathfrak{p}^m$, which is true if and only if $\mathrm{ord}_{\mathfrak{p}}(y+z) \geq m$. Similarly if $m < 0$ then consider $t \in \mathfrak{p} - \mathfrak{p}^2$ and let $y' := t^{-m}y, z' := t^{-m}z$ so that $\mathrm{ord}_{\mathfrak{p}}(y' + z') \geq 0$. Now recall that $y' + z' = t^{-m}(y + z)$ so

$$\mathrm{ord}_{\mathfrak{p}}(y' + z') = -m + \mathrm{ord}_{\mathfrak{p}}(y + z) \geq 0$$

becomes

$$\mathrm{ord}_{\mathfrak{p}}(y + z) \geq m = \min\{\mathrm{ord}_{\mathfrak{p}}(y), \mathrm{ord}_{\mathfrak{p}}(z)\}.$$

Therefore given a prime $\mathfrak{p}$ we have a discrete valuation $\mathrm{ord}_{\mathfrak{p}}$ on $K^\times$. We know $\mathrm{ord}_{\mathfrak{p}}(F^\times) = 0$ because $F$ is a subring of $R_x$, so $\mathfrak{p} \cap F$ is an ideal of $F$ which is either prime or zero. We know that since $F$ is a field, the only ideals of $F$ are $(0)$ and $F$, and since $\mathfrak{p}$ does not contain 1, we know $F \cap \mathfrak{p} = (0)$ so for any $a \in F$, $\mathrm{ord}_{\mathfrak{p}}(a) = 0$.

Consider the discrete valuation ring $\mathrm{O}_{\mathfrak{p}}$ of $\mathrm{ord}_{\mathfrak{p}}$. Since $(R_x)_{\mathfrak{p}}$ is a local ring with unique maximal ideal $\mathfrak{p}(R_x)_{\mathfrak{p}}$, $R_x \cap \mathfrak{p}(R_x)_{\mathfrak{p}} = \mathfrak{p}$. Therefore if we can show $\mathrm{O}_{\mathfrak{p}} = (R_x)_{\mathfrak{p}}$ we will have completed our bijection.

Since $\mathfrak{p} \subset \mathrm{O}_{\mathfrak{p}}$, $(R_x - \mathfrak{p}) \subset \mathrm{O}_{\mathfrak{p}}^\times$ we know $(R_x)_{\mathfrak{p}} \subset \mathrm{O}_{\mathfrak{p}}$. Meanwhile if $f \in \mathrm{O}_{\mathfrak{p}}$, then if $t \in \mathfrak{p} - \mathfrak{p}^2$, $f = t^{\mathrm{ord}_{\mathfrak{p}}(f)}(t^{-\mathrm{ord}_{\mathfrak{p}}(f)}f)$. Since $t \in R_x$ and $\mathrm{ord}_{\mathfrak{p}}(t^{-\mathrm{ord}_{\mathfrak{p}}(f)}f) = 0$, if we can show that $\mathrm{O}_{\mathfrak{p}}^\times \subset (R_x)_{\mathfrak{p}}$ then we will have our result.

So consider $f \in \mathrm{O}_{\mathfrak{p}}^\times$. Since $\mathrm{O}_{\mathfrak{p}} \subset K$ and $K$ is the fraction field of $R_x$, we may find $y, z \in R_x - \{0\}$ such that $f = y/z$. Since $\mathrm{ord}_{\mathfrak{p}}(y/z) = 0$,

$\mathrm{ord}_{\mathfrak{p}}(y) = \mathrm{ord}_{\mathfrak{p}}(z)$. If we consider the fractional ideal $f = (y)(z)^{-1}$, then $(f) = \mathfrak{a}\mathfrak{b}^{-1}$ where $(\mathfrak{a}, \mathfrak{b}) = R_x$ and both $\mathfrak{a}$ and $\mathfrak{b}$ are coprime to $\mathfrak{p}$. Let $\mathfrak{q}_1^{e_1} \dots \mathfrak{q}_m^{e_m}$ be the prime ideal factorization of $\mathfrak{a}$. If we pick $r \in R_x - \mathfrak{p}$ then by the chinese remainer theorem we may find $a \in R_x$ such that $a \equiv r \bmod \mathfrak{p}$ and $a \equiv 0 \bmod \mathfrak{q}_i^{e_i}$ for all $1 \leq i \leq m$. Therefore $a \in \mathfrak{a}$, so $(a) \subset \mathfrak{a}$ and since $(a)$ has a prime ideal factorization, $(a) = \mathfrak{a}\mathfrak{c}$ where $(\mathfrak{c}, \mathfrak{p}) = R_x$.

Since $(f) = \mathfrak{a}\mathfrak{b}^{-1}$, $f\mathfrak{b} = \mathfrak{a}$ and so

$$f\mathfrak{b}\mathfrak{c} = \mathfrak{a}\mathfrak{c} = (a).$$

Therefore $\mathfrak{b}\mathfrak{c} = (a/f)$, if we let $a/f = b$ then $b \in R_x - \mathfrak{p}$ because $\mathfrak{b}\mathfrak{c} \subset R_x$ and both $\mathfrak{b}$ and $\mathfrak{c}$ are coprime to $\mathfrak{p}$. Therefore

$$f = ab^{-1} \in R_x(R_x - \mathfrak{p})^{-1} = (R_x)_{\mathfrak{p}},$$

and $\mathrm{O}_{\mathfrak{p}} = (R_x)_{\mathfrak{p}}$.

□

We will use Theorem 2.12 to prove a vital fact about the zeros and poles of elements of $K^\times$. For this result to make sense however, we must first prove a lemma relating our field of constants, $F$ to our valuation ring $\mathrm{O}_P$.

**Lemma 2.13.** *For any place $P$, $[\mathrm{O}_P / \mathfrak{m}_P : F]$ is finite. In particular, if $F$ is algebraically closed, $\mathrm{O}_P / \mathfrak{m}_P \cong F$.*

*Proof.* Since $\mathrm{ord}_P$ is trivial on $F^\times$, $F^\times \subset \mathrm{O}_P^\times$. Moreover $F \hookrightarrow \mathrm{O}_P / \mathfrak{m}_P$ because any nontrivial ring homomorphism out of a field is injective. We will show that $\mathrm{O}_P / \mathfrak{m}_P$ is a finite extension of $F$.

Fix $x \in K - F$ with $\mathrm{ord}_P(x) \geq 1$. The extension $K/F(x)$ is then finite. Let $\{e_1, \dots, e_m\}$ be elements of $\mathrm{O}_P$ whose residue classes modulo $\mathfrak{m}_P$ are linearly independent over $F$. We will show that $m \leq [K : F(x)]$. Assume to the contrary that $m > [K : F(x)]$. Then we have a set of rational functions $\{f_1(x), \dots, f_m(x)\} \subset F(x)$, not all zero, giving a linear relation

$$f_1(x)e_1 + \cdots + f_m(x)e_m = 0.$$

By clearing denominators, we can take the $f_i(x)$ to be elements of $F[x] \subset \mathrm{O}_P$, not all zero. If none of the $f_i$ have a nonzero constant term, divide both sides of the equation by $x$ until at least one $f_i$ has a nonzero constant term. Let $c_i$ be the constant term of $f_i(x)$. Recalling that $\mathrm{ord}_P(x) \geq 1$, when we reduce modulo $\mathfrak{m}_P$, all the $x$'s vanish and we have the linear relation

$$c_1\overline{e_1} + \cdots + c_m\overline{e_m} = 0$$

in $\mathrm{O}_P / \mathfrak{m}_P$. This is in contradiction to the linear independence of the $\overline{e_i}$'s over $F$, so $m \leq [K : F(x)]$.

□

**Theorem 2.14.** *For any $x \in K - F$,*

$$[K : F(x)] = \sum_P \max(\mathrm{ord}_P(x), 0)[\mathrm{O}_P / \mathfrak{m}_P : F].$$

*Proof.* Let $x \in K - F$ and let $R_x$ be the integral closure of $F[x]$ in $K$. As mentioned in the proof of Theorem 2.12, $R_x$ is a Dedekind domain. Therefore $xR_x$ factors uniquely as

$$xR_x = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(x)}.$$

So by the chinese remainder theorem,

$$R_x/xR_x = R_x/\prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(x)} \cong \prod_{\mathfrak{p}} R/\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(x)}.$$

Since $R_x$ is the integral closure of $F[x]$ in a finite field extension, it is a free $F[x]$ module of rank $[K : F(x)]$. Let $\{e_1, \dots, e_{[K:F(x)]}\} \subset R_x$ so that $R_x = \bigoplus_j F[x]e_j$. Therefore $xR_x = \bigoplus_j xF[x]e_j$ and

$$R_x/xR_x = \bigoplus_j (F[x]/xF[x])\, e_j \cong \bigoplus_j Fe_j.$$

This tells us that $\dim_F(R_x/xR_x) = [K : F(x)]$. We wish to show that $\dim_F \prod_{\mathfrak{p}} R_x/\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(x)} = \sum_P \max\{\mathrm{ord}_P(x), 0\}[\mathrm{O}_P/\mathfrak{m}_P : F]$. We can do this if we briefly assume that $\dim_F(R_x/\mathfrak{p}^k)$ is finite for all $k \geq 0$. Since $\mathrm{ord}_{\mathfrak{p}}(x) = 0$ for all but finitely many $\mathfrak{p}$, $\prod_{\mathfrak{p}} R_x/\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(x)}$ is a finite product, so

$$\dim_F(\prod_{\mathfrak{p}} R_x/\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(x)}) = \sum_{\mathfrak{p}} \dim_F(R_x/\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(x)}).$$

We know from the properties of rings that $R_x/\mathfrak{p}^{k-1} \cong (R_x/\mathfrak{p}^k)/(\mathfrak{p}^{k-1}/\mathfrak{p}^k)$, so

$$\dim_F(R_x/\mathfrak{p}^k) = \dim_F(R_x/\mathfrak{p}^{k-1}) + \dim_F(\mathfrak{p}^{k-1}/\mathfrak{p}^k) = \sum_{j=1}^{k} \dim_F(\mathfrak{p}^{j-1}/\mathfrak{p}^j).$$

To make use of this, consider that for $t \in \mathfrak{p} - \mathfrak{p}^2$, the map $R_x \to \mathfrak{p}^{j-1}/\mathfrak{p}^j$ by $f \mapsto t^{j-1}f \bmod \mathfrak{p}^j$ is surjective. The kernel will be $\{f \in R_x | t^{j-1}f \in \mathfrak{p}^j\}$ i.e. $\{f | f \in t^{1-j}\mathfrak{p}^j = \mathfrak{p}\}$. Therefore

$$\sum_{j=1}^{k} \dim_F(\mathfrak{p}^{j-1}/\mathfrak{p}^j) = \sum_{j=1}^{k} \dim_F(R_x/\mathfrak{p}) = k \dim_F(R_x/\mathfrak{p}).$$

Therefore, if we know that $\dim_F(R_x/\mathfrak{p})$ is finite,

$$\sum_{\mathfrak{p}} \dim_F(R_x/\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(x)}) = \sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(x) \dim_F(R_x/\mathfrak{p})$$

.

To determine the dimension, consider the map $R_x \to (R_x)_{\mathfrak{p}}/\mathfrak{p}(R_x)_{\mathfrak{p}}$ defined by $f \mapsto f/1 \bmod \mathfrak{p}(R_x)_{\mathfrak{p}}$. This map is onto and the kernel of the map is $\mathfrak{p}$, as we show here. The map is onto because $R_x$ is Dedekind, so every prime ideal of $R_x$ is maximal and so if $z \in R_x - \mathfrak{p}$ we can find $z' \in R_x$ so that $zz' \equiv 1 \bmod \mathfrak{p}$. This is enough to show that $f \mapsto f/1 \bmod \mathfrak{p}(R_x)_{\mathfrak{p}}$ is onto because for any $y \in R_x$,h $z \in R_x - \mathfrak{p}$, we can take $f = yz'$. Because $fz \cong y \bmod \mathfrak{p}$, $f \equiv y/z \bmod \mathfrak{p}(R_x)_\mathfrak{p}$. Therefore

$$\sum_{\mathfrak{p}} \operatorname{ord}_\mathfrak{p}(x) \dim_F(R_x/\mathfrak{p}) = \sum_{\mathfrak{p}} \operatorname{ord}_\mathfrak{p}(x) \dim_F((R_x)_\mathfrak{p}/\mathfrak{p}(R_x)_\mathfrak{p}).$$

Now by Theorem 2.12, to each prime $\mathfrak{p}$ we have a unique valuation $P$ such that $\operatorname{ord}_P(x) \geq 0$ and each valuation is given by a prime. Also by Theorem 2.12, the localization of $R_x$ at $\mathfrak{p}$ is the valuation ring $\mathcal{O}_P$, so $\mathcal{O}_P/\mathfrak{m}_P = (R_x)_\mathfrak{p}/\mathfrak{p}(R_x)_\mathfrak{p}$ and

$$\sum_{\mathfrak{p}} \operatorname{ord}_\mathfrak{p}(x) \dim_F((R_x)_\mathfrak{p}/\mathfrak{p}(R_x)_\mathfrak{p}) = \sum_{P:\operatorname{ord}_P(x)\geq 0} \operatorname{ord}_P(x) \dim_F(\mathcal{O}_P/\mathfrak{m}_P).$$

We know by Lemma 2.13 that $\dim_F(\mathcal{O}_P/\mathfrak{m}_P) = [\mathcal{O}_P/\mathfrak{m}_P : F]$ is finite for all $P$, so $[K : F(x)] = \sum_{P:\operatorname{ord}_P(x)\geq 0} \operatorname{ord}_P(x)[\mathcal{O}_P/\mathfrak{m}_P]$ and our assertion is proved.

□

We now know the basic structure of function fields, the objects we will study to prove the Riemann-Roch Theorem. The reason they are called function fields in general should now be obvious. When $F$ is algebraically closed, we may interpret any $f \in K$ as an $F$-valued function defined on all but finitely many places $P$ by $f(P) = a$ if $\operatorname{ord}_P(f) \geq 0$ and $a$ is the element of $F$ where $a \equiv f \bmod \mathfrak{m}_P$ i.e. $\operatorname{ord}_P(f - a) \geq 1$.

This resembles the interpretation of rational functions in $\mathbb{C}(z)$ as $\mathbb{C}$-valued functions on all but finitely many points of the Riemann Sphere. For this reason, it is not totally out of the question to think about the elements of $K$ as analogues of meromorphic functions on a curve. Note that if $F$ were not algebraically closed, the value would lie not in $F$, but instead in the typically larger field $\mathcal{O}_P/\mathfrak{m}_P$.

**Example 2.15.** If $F = \mathbb{R}$, $K = \mathbb{R}(x)$ and $P = \operatorname{ord}_{x^2+1}$, then $\mathcal{O}_P/\mathfrak{m}_P \cong \mathbb{C}$ via $x \mapsto i$. For instance if $f = 2x^2 + 3/x$, then $f(P) = 2(i^2) + 3/i = -2 - 3i$.

Now because we know that for any $x \in K^\times$, $\operatorname{ord}_P(x) = 0$ for all but finitely many places $P$. We can now introduce an important book-keeping object of our study: the group of divisors.

*Convention*: To make things simpler, our field of constants, $F$, will be algebraically closed from here on unless otherwise noted.

## 3. Divisors

**Definition 3.1.** The group of divisors of a function field $K/F$, $\mathcal{D}_K$, is the free abelian group on the places of $K$. That is,

$$\mathcal{D}_K = \bigoplus_P \mathbb{Z}P = \left\{ \sum_P n_P P : n_P = 0 \text{ for all but finitely many } P \right\}.$$

Here the group operation is defined componentwise, so that $\sum_P n_P P + \sum_P m_P P = \sum_P (n_P + m_P)P$.

**Example 3.2.** For any function field $K$, the *zero divisor* of $K$ is $\mathbf{0} = \sum_P n_P P$ where $n_P = 0$ for all $P$.

**Example 3.3.** For a rational function field $F(x)$, we identify the places of $F(x)/F$ with elements of $F \cup \{\infty\}$, because as we saw in Example 2.11, $\mathrm{ord}_P$ is either $\mathrm{ord}_a$ for $a \in F$ or $\mathrm{ord}_\infty$. If we let $F = \mathbb{C}$, $4(i) - 1(\infty)$ is an example of a divisor on $F(x)/F$.

**Example 3.4.** For a nonexample of a divisor consider a function field $K/F$ and $\sum_P n_P P$ where $n_P = 1$ for all $P$. This is not a divisor because it has nonzero coefficients at an infinite number of places $P$.

**Definition 3.5.** The *degree* of a divisor is $\deg(\sum_P n_P P) = \sum_P n_P \in \mathbb{Z}$. The *support* of a divisor, $\mathrm{supp}(\sum_P n_P P)$, is the set of places where $n_P \neq 0$.

**Example 3.6.** Let a divisor $D$ in $\mathbb{C}(z)$ be defined as $(i) + (-i) - 2(1)$. The degree is $\deg(D) = 1 + 1 - 2 = 0$. Notice that for $f(z) = \frac{(z-i)(z+i)}{(z-1)^2}$, $D = \sum_P \mathrm{ord}_P(f)P$. This sort of divisor is very important for our study.

**Remark 3.7.** The definition of the degree is well-defined because in all cases it is a finite sum by Theorem 2.14. Also, if we allowed function fields to be defined over a non-algebraically closed field, we would need to let the degree of $\sum_P n_P P$ be $\sum_P n_P [\mathrm{O}_P/P : F]$ as one might imagine from Theorem 2.14 and Corollary 3.10.

**Definition 3.8.** For any $f \in K^\times$ we will define the divisor of $f$, $\mathrm{div}(f)$ to be $\sum_P \mathrm{ord}_P(f)P$. Note that this is well-defined by Theorem 2.14. We can in fact show more.

**Definition 3.9.** For any $f \in K$ we can split up the divisor $\mathrm{div}(f)$ into its positive and negative components as

$$\mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f),$$

where

$$\mathrm{div}_0(f) = \sum_P \max(\mathrm{ord}_P(f), 0)P,$$

and

$$\mathrm{div}_\infty(f) = \sum_P - \min(\mathrm{ord}_P(f), 0).$$

We frequently call $\mathrm{div}_0(f)$ the *divisor of zeroes* of $f$ and $\mathrm{div}_\infty(f)$ the *divisor of poles* of $f$ for reasons already discussed.

**Corollary 3.10.** *For any $x \in K - F$,*

$$\deg(\mathrm{div}_0(x)) = \deg(\mathrm{div}_\infty(x)) = [K : F(x)].$$

*Hence when we count multiplicities, for any $x \in K - F$ there are as many zeros of $x$ as there are poles.*

*Proof.* By Theorem 2.14, $\deg(\mathrm{div}_0(x)) = [K : F(x)]$. Now if we replace $x$ with $1/x$, since $F(x) = F(1/x)$ and $\mathrm{div}_0(1/x) = \mathrm{div}_\infty(x)$, $\deg(\mathrm{div}_\infty(x)) = [K : F(x)]$. □

**Theorem 3.11.** *The following properties hold for arbitrary divisors $E, D \in \mathcal{D}_K$ and all $f, g \in K^\times$:*

- $\deg(E + D) = \deg(E) + \deg(D)$,
- $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$,
- $\deg(\mathrm{div}(f)) = 0$.

*Proof.* It is simple computation to show that for any divisors $E$ and $D$, $\deg(E + D) = \deg(E) + \deg(D)$.

To show $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$, recall that $\mathrm{ord}_P(\cdot)$ is a valuation on $K$. Thus for all $f, g \in K^\times$, $\mathrm{ord}_P(fg) = \mathrm{ord}_P(f) + \mathrm{ord}_P(g)$ for all places $P$. Thus $\sum_P \mathrm{ord}_P(fg) = \sum_P(\mathrm{ord}_P(f) + \mathrm{ord}_P(g))$.

Finally, $\mathrm{div}(a) = \mathbf{0}$ for all $a \in F^\times$ so $\deg(\mathrm{div}(a)) = \sum_P 0 = 0$. For $x \in K - F$, $\deg(\mathrm{div}(x)) = \deg(\mathrm{div}_0(x)) - \deg(\mathrm{div}_\infty(x)) = 0$.

□

**Corollary 3.12.** *For any divisor $D$, $f \in K^\times$, $\deg(D + \mathrm{div}(f)) = \deg(D)$.*

*Proof.* We know by the above that $\deg(D+\mathrm{div}(f)) = \deg(D)+\deg(\mathrm{div}(f))$ and $\deg(\mathrm{div}(f)) = 0$.

□

**Definition 3.13.** We'll define a partial order on $\mathcal{D}_K$ by saying that

$$\sum_P n_P P \geq \sum_P m_P P \iff n_P \geq m_P \text{ for all } P,$$

in the standard order on the integers.

**Example 3.14.** It is easy enough to check that for

$$f(x) = (x - i)^4, \ D = -4(i) + 5(\infty),$$

on $K = \mathbb{C}(x)$ that $\mathrm{div}(f) = 4i - 4\infty$ and $\mathrm{div}(f) + D \geq \mathbf{0}$.

**Definition 3.15.** Let

$$L(D) := \left\{f \in K^\times : \mathrm{div}(f) + D \geq \mathbf{0}\right\} \cup \{0\}.$$

We include 0 in $L(D)$ so that $L(D)$ is an $F$-vector space. When $L(D)$ is finite-dimensional over $F$, set

$$\ell(D) := \dim_F(L(D)).$$

The space of functions referred to in the introduction is $L(D)$ and the conditions on the zeros and poles of a function will be described by the divisor $D$. Theorem 3.20 below shows that $L(D)$ is finite-dimensional over $F$ for any divisor $D$. This is a major result because the purpose of the Riemann-Roch theorem is to compute $\ell(D)$.

**Definition 3.16.** A divisor $\sum_P n_P P$ is called *effective at* $P$ if $n_P \geq 0$. A divisor is called *effective* if it is effective at each point $P$.

**Example 3.17.** For all $x \in K - F$, $\operatorname{div}_0(x)$ and $\operatorname{div}_\infty(x)$ are effective by definition.

**Remark 3.18.** For any divisor $D = \sum_P n_P P$, $f \in L(D)$ if and only if $\operatorname{ord}_P(f) \geq -n_P$ for all $P$. Riemann was concerned with the existence of functions with poles at proscribed places and bounds on the orders of such poles. If we consider only effective divisors $D = \sum_P n_P P$ as Riemann did, $L(D)$ is the space of functions with a pole at $P$ of order at most $n_P$ for all $P$. If we were to consider a divisor in the more general sense, say $-4(i) + 2(3) + 5(\infty)$ on $\mathbb{C}(x)$, then $L(D)$ would be the space of functions with a zero of at least order 4 at $i$ and poles of at most order 2 at 3 and at most order 5 at $\infty$.

**Corollary 3.19.** *For a divisor $D$ where $\dim_F(L(D)) < \infty$ and any element $g \in K$, $L(D + \operatorname{div}(g)) = L(D)$. Thus $\ell(D + \operatorname{div}(g))$ is defined and equal to $\ell(D)$.*

*Proof.* If $L(D) = \{f \in K : \operatorname{div}(f) + D \geq \mathbf{0}\}$ then

$$\begin{aligned} L(D + \operatorname{div}(g)) &= \left\{f \in K^\times : \operatorname{div}(f) + \operatorname{div}(g) + D \geq \mathbf{0}\right\} \cup \{0\} \\ &= \left\{f \in K^\times : \operatorname{div}(fg) + D \geq \mathbf{0}\right\} \cup \{0\} \\ &= \left\{h \in K^\times : \operatorname{div}(h) + D \geq \mathbf{0}\right\} \cup \{0\} \\ &= L(D). \end{aligned}$$

□

**Theorem 3.20.** *Given any $D$ for which $\ell(D)$ is defined, $\ell(D+P) \leq \ell(D)+1$ for all points $P$.*

*Proof.* Let $D = n_P P + \sum_{Q \neq P} n_Q Q$ be a divisor for which $L(D)$ is finite-dimensional over $F$. Then for all $f \in L(D+P)$ we have $\operatorname{ord}_P(f) \geq -n_P - 1$. If additionally $f \notin L(D)$ then we know that for some place $Q$, $\operatorname{ord}_Q(f) + n_Q < 0$. Since $f \in L(D+P)$, this can only happen for $Q = P$ and $\operatorname{ord}_P(f) + n_P = -1$. Let $m = n_P + 1$ so that $\operatorname{ord}_P(f) = -n_P - 1 = -m$. Therefore any $f \in L(D+P) - L(D)$ has exact order $-m$ at $P$. If such an $f$ does not exist, then $L(D+P) = L(D)$.

If we have an $f$ in $L(D+P)$ of exact order $-m$ at $P$ then consider some uniformizing parameter $t$ of $P$. Since $\operatorname{ord}_P(f) = -m$ and $\operatorname{ord}_P(t^m) = m$, $t^m f \in \mathrm{O}_P - \mathfrak{m}_P$. Since $\mathrm{O}_P / \mathfrak{m}_P \cong F$, $t^m f \equiv a \bmod P$ for some $a \in F^\times$ and thus $t^m f = a + xt$ for some $x \in \mathrm{O}_P$. If $g$ is some other element of $L(D+P)$ of exact order $-m$ at $P$ then we can likewise write $t^m g = b + yt$ for $b \in F$, $y \in \mathrm{O}_P$. Thus

$$\begin{aligned} f &= at^{-m} + xt^{-m+1}, \\ g &= bt^{-m} + yt^{-m+1}, \end{aligned}$$

so

$$g - \frac{b}{a} f = (y - \frac{b}{a} x) t^{-m+1}.$$

Note here that

$$\mathrm{ord}_P(g - \frac{b}{a} f) = \mathrm{ord}_P(y - \frac{b}{a} x) + \mathrm{ord}_P(t^{-m+1}) \geq -m + 1 = -n_P$$

and so between any two nonzero elements of $L(D+P)/L(D)$ we have a linear dependence. Thus $\dim_F L(D+P)/L(D) = 1$ so $\ell(D+P) \leq 1 + \ell(D)$. □

This theorem gives us the tools to show not only that $\ell(D)$ is finite, but to put a sharp upper bound on $\ell(D)$.

**Corollary 3.21.** *For any divisor $D$ of $K$, either $\ell(D) \leq \deg(D) + 1$ or $\ell(D) = 0$.*

*Proof.* We first prove that if $\deg(D) < 0$ then $\ell(D) = 0$. To show this, suppose to the contrary that there exists a nonzero $g \in L(D)$. Since $g \in L(D)$, $\deg(\mathrm{div}(g) + D) \geq 0$ by the definition of $L(D)$, but on the other hand we already know that $\deg(\mathrm{div}(g) + D) = \deg(D) < 0$. Thus there is no nonzero $g \in L(D)$, so $\ell(D) = 0$ when $\deg(D) < 0$.

If $\deg(D) = 0$ then for any point $P$ let $D_P = D - P$, so $\deg(D_P) = -1$. Since $\ell(D_P) = 0$, $\ell(D) = \ell(D_P + P) \leq \ell(D_P) + 1 = 1$.

Suppose by induction that for any divisor $E$ with $\deg(E) = n \geq 0$ that $\ell(E) \leq n+1$. Consider any divisor $D$ of degree $n+1$. If we set $D_P = D - P$ then

$$\ell(D) = \ell(D_P + P) \leq \ell(D_P) + 1 \leq (n+1) + 1 = \deg(D) + 1.$$

□

**Corollary 3.22.** *The bound $\ell(D) \leq \deg(D) + 1$ is sharp for any function field $K$.*

*Proof.* For any function field $K$, consider the zero divisor $\mathbf{0}$. By Theorem 2.14, if $f \in K$ has no poles, it has no zeros and therefore $f \in F$. Therefore $\ell(\mathbf{0}) = 1$. □

## 4. The Adeles and Riemann's Inequality

From our last section we have an upper bound on $\ell(D)$ depending on the degree of $D$. In this section we wish to show that we also have a lower bound depending on the degree of $D$. In particular, we wish to show that there is a constant $g$ depending on $K$ so that $\ell(D) \leq \deg(D) - g + 1$ and that this inequality is sharp. To this end, we introduce the adele space of $K$. Its name comes from the adeles of a number field $\Bbbk$. Consider the direct product $\prod_v \Bbbk$ indexed by the absolute values $|\cdot|_v$ of $\Bbbk$. The adeles of a number field are a subset of that direct product where all but finitely many coordinates will be inside of $\mathrm{O}_v$, a distinguished subring of $\Bbbk$. The adeles are often called the "restricted direct product" of $\Bbbk$ with respect to $\mathrm{O}_v$.

**Definition 4.1.** The *adele ring* $\mathbb{A}_K$ of a function field $K$ is the restricted direct product of $K$ with respect to $\mathrm{O}_P$ indexed by the places $P$ of $K$. Elements of the adeles will be of the form $\prod_P x_P$ which we will often write as $(x_P)$.

The *diagonal embedding* is the map $x \mapsto (x, x, x, \ldots)$, which is in $\mathbb{A}_K$ by Theorem 2.14.

The *adele space* $A_K(D)$ associated to a divisor $D = \sum_P n_P P$ is the set of all adeles $(x_P)$ where $\mathrm{ord}_P(x_P) + n_P \geq 0$ or $x_P = 0$. Because $n_P = 0$ for all but finitely many $P$, $A_K(D) \subset \mathbb{A}_K$ for all $D$.

**Example 4.2.** Taking $K = \mathbb{C}(x)$, some elements of $A_K((1) - 5(\infty) + 3(i))$ are $(x-1)(x-i)^4$ under the diagonal embedding and the adele $(y_P)$ where $y_P = 0$ for all $P \neq -\deg(\cdot) = \mathrm{ord}_\infty$ and for $y_\infty = x^n$ where $n \geq 5$.

**Remark 4.3.** Traditionally when talking about the adeles of a number field $\Bbbk$, one uses the product of the completions of $\Bbbk$ with respect to the metric induced by the absolute value $|\cdot|_v$. From a place of $P$, we can induce a metric $|\cdot|_P$ and complete $K$ with respect to that metric. For this reason, many proofs of the Riemann-Roch Theorem define the adeles to be the restricted direct product over $P$ of $K_P$ with respect to $\boldsymbol{\mathcal{O}}_P$ where $K_P$ is the completion of $K$ with respect to $|\cdot|_P$ and $\boldsymbol{\mathcal{O}}_P$ is the discrete valuation ring of $K_P$.

**Lemma 4.4.** *The following properties hold for $A_K(\cdot)$ where $D = \sum_P n_P P$ and $E = \sum_P m_P P$:*

- *If $D \leq E$ then $A_K(D) \subset A_K(E)$.*
- *If we let $\min\{D, E)\} = \sum_P \min\{n_P, m_P\} P$ then $A_K(\min\{D, E\}) = A_K(D) \cap A_K(E)$*
- *If we let $\max\{D, E\} = \sum_P \max\{n_P, m_P\} P$, then $A_K(\max\{D, E\}) = A_K(D) + A_K(E)$.*
- *Under the diagonal embedding, $K \cap A_K(D) = L(D)$.*

*Proof.* Let $D = \sum_P n_P P$ and $E = \sum_P m_P P$.

If $E \geq D$ then $m_P \geq n_P$ for all $P$. If $(\phi_P) \in A_K(D)$ then for all $P$ so that $\phi_P \neq 0$, $\mathrm{ord}_P(\phi_P) + m_P \geq \mathrm{ord}_P(\phi_P) + n_P \geq 0$.

For this reason, $A_K(\min(D, E)) \subset A_K(D) \cap A_K(E)$. If $(\phi_P) \in A_K(D)$ and $(\phi_P) \in A_K(E)$ then for all $P$ so that $\phi_P \neq 0$, $\mathrm{ord}_P(\phi_P) + n_P \geq 0$ and $\mathrm{ord}_P(\phi_P) + m_P \geq 0$. Therefore $\mathrm{ord}_P(\phi_P) + \min\{n_P, m_P\} \geq 0$, so

$$A_K(D) \cap A_K(E) = A_K(\min\{D, E\}).$$

If $(\phi_P) \in A_K(D)$ and $(\psi_P) \in A_K(E)$, then for the places $P$ where $\phi_P = -\psi_P$ there's nothing to show. If at least one of $\phi_P$, $\psi_P$ is nonzero, then we introduce the convention that if $\phi_P$ is zero and $\psi_P$ is nonzero, then $\min\{\mathrm{ord}_P(\phi_P), \mathrm{ord}_P(\psi_P)\} = \mathrm{ord}_P(\psi_P)$.

With this in mind, $\mathrm{ord}_P(\phi_P + \psi_P) \geq \min\{\mathrm{ord}_P(\phi_P), \mathrm{ord}_P(\psi_P)\}$ by the definition of a valuation. Therefore for all places $P$,

$$\mathrm{ord}_P(\phi_P + \psi_P) + \max\{n_P, m_P\} \geq \min\{\mathrm{ord}_P(\phi_P), \mathrm{ord}_P(\psi_P)\} + \max\{n_P, m_P\}$$

and

$$\min\{\mathrm{ord}_P(\phi_P), \mathrm{ord}_P(\psi_P)\} + \max\{n_P, m_P\} \geq 0.$$

The last assertion follows by the definitions of $A_K(D)$ and $L(D)$.

□

**Lemma 4.5.** *If $D \leq E$ as divisors then* $\dim_F\left(\frac{A_K(E)}{A_K(D)}\right) = \deg(E) - \deg(D)$.

*Proof.* We induct on $\deg(E) - \deg(D)$ to achieve this result. If $\deg(E) - \deg(D) = 0$ then since $D \leq E$ we must have $D = E$ and so $A_K(E) = A_K(D)$ and $\frac{A_K(E)}{A_K(D)} = \{0\}$.

If $\deg(E) - \deg(D) = 1$ and $D \leq E$ then $E = D + P$ for some place $P$. So showing the theorem holds if $\deg(E) - \deg(D) = 1$ is the same as showing that $\dim_F\left(\frac{A_K(D+P)}{A_K(D)}\right) = 1$ for all divisors $D$ and all places $P$. If we take an arbitrary divisor $D = \sum_P n_P P$ and place $P$, then we can project from $A_K(D+P)$ to $\mathfrak{m}_P^{-n_P-1}$ and then reduce modulo $\mathfrak{m}_P^{-n_P}$ giving the map $(x_Q) \mapsto x_P \bmod \mathfrak{m}_P^{-n_P}$.

This map is onto because if $(x_Q) \in A_K(D)$ then for any $f \in \mathfrak{m}_P^{-n_P-1}$, $f \times \prod_{Q \neq P} x_Q \in A_K(D+P)$.

If $(x_Q) \in A_K(D+P)$ is in the kernel of this map, then $x_P \in \mathfrak{m}_P^{-n_P}$ so $\mathrm{ord}_P(x_P) \geq -n_P$ or $x_P = 0$. Since $(x_Q) \in A_K(D+P)$, for $Q \neq P$, $\mathrm{ord}_Q(x_Q) \geq -n_Q$ and $(x_Q) \in A_K(D)$.

Therefore $A_K(D+P)/A_K(D) \cong \mathfrak{m}_P^{-n_P-1}/\mathfrak{m}^{-n_P}$. Our theorem then holds when $\deg(E) - \deg(D) = 1$ because $\mathfrak{m}_P^k/\mathfrak{m}_P^{k+1} \cong \mathrm{O}_P/\mathfrak{m}_P$. To see this, let $t$ be a uniformizing parameter of $P$ and consider the map $\mathrm{O}_P \to \mathfrak{m}_P^k/\mathfrak{m}_P^{k+1}$ by $f \to ft^k \bmod \mathfrak{m}_P^{k+1}$. This map is onto with kernel $\mathfrak{m}_P$.

If for some $n \geq 1$, $\dim_F\left(\frac{A_K(E)}{A_K(D)}\right) = \deg(E) - \deg(D)$ for all $E \geq D$ with $\deg(E) - \deg(D) = n$, then we can find some divisor $E'$ where $E \geq E' \geq D$ satisfying

$$\deg(E) - \deg(E') = 1,$$

and

$$\deg(E') - \deg(D) = n.$$

Since $A_K(D) \subset A_K(E') \subset A_K(E)$,

$$\begin{aligned}
\dim_F\left(\frac{A_K(E)}{A_K(D)}\right) &= \dim_F\left(\frac{A_K(E)}{A_K(E')}\right) + \dim_F\left(\frac{A_K(E')}{A_K(D)}\right) \\
&= \deg(E) - \deg(E') + \deg(E') - \deg(D) \\
&= \deg(E) - \deg(D)
\end{aligned}$$

□

**Lemma 4.6.** *If $D \leq E$ are divisors of $K$ then when we view $K$ diagonally in $F$,*

$$\dim_F \frac{A_K(E)+K}{A_K(D)+K} = (\deg(E) - \ell(E)) - (\deg(D) - \ell(D)).$$

*Proof.* Because the map $A_K(E) \to A_K(E) + K \to \frac{A_K(E)+K}{A_K(D)+K}$ is onto with kernel $A_K(E) \cap (A_K(D) + K)$,

$$\frac{A_K(E)+K}{A_K(D)+K} \cong \frac{A_K(E)}{A_K(E) \cap (A_K(D)+K)} = \frac{A_K(E)}{A_K(D)+L(E)}.$$

Furthermore, $\frac{A_K(E)}{A_K(D)+L(E)} \cong \frac{A_K(E)/A_K(D)}{(A_K(D)+L(E))/A_K(D)}$ so that

$$\dim_F \left(\frac{A_K(E)+K}{A_K(D)+K}\right) = \deg(E) - \deg(D) - \dim_F \left(\frac{A_K(D)+L(E)}{A_K(D)}\right).$$

Now since $\frac{A_K(D)+L(E)}{A_K(D)} \cong \frac{L(E)}{A_K(D) \cap L(E)}$ we would like $A_K(D) \cap L(E) = L(D)$. This is no problem because $K \cap A_K(D) = L(D)$ and $L(E) \subset K$ so $A_K(D) \cap L(E) \subset L(D)$. But $L(D) \subset L(E)$ because $E \geq D$ so we have equality. Thus $\frac{(A_K(D)+L(E))}{A_K(D)} \cong \frac{L(E)}{L(D)}$ and since $\dim_F(\frac{L(E)}{L(D)}) = \ell(E) - \ell(D)$ our assertion is proved. □

**Definition 4.7.** For a divisor $D$, set $r(D) = \deg(D) - \ell(D)$.

**Example 4.8.** For the zero divisor $\mathbf{0}$, $r(\mathbf{0}) = \deg(\mathbf{0}) - \ell(\mathbf{0}) = 0 - 1 = -1$.

**Lemma 4.9.** *If $f \in K^\times$ and $E, D$ are divisors on $K$, then the function $r : \mathcal{D}_K \to \mathbb{Z}$ satisfies the following:*

- *if $D \leq E$ then $r(D) \leq r(E)$,*
- *for any $D$, $r(\operatorname{div}(f) + D) = r(D)$.*

*Proof.* The second assertion is a consequence of the properties of the degree of a divisor and $\ell(E)$: $\deg(\operatorname{div}(f)+D) = \deg(D)$ and $\ell(\operatorname{div}(f)+D) = \ell(D)$. The first assertion follows from Lemma 4.6. □

**Theorem 4.10.** *If $K$ is a function field over $F$, $r(D)$ is bounded above for all divisors $D$.*

*Proof.* Pick an arbitrary $x \in K - F$. By Corollary 3.10, $\deg(\operatorname{div}_\infty(x)) = [K : F(x)]$ which we call $n$ for brevity. Consider an arbitrary $y \in R_x$, the integral closure of $F[x]$ in $K$. If $\operatorname{ord}_P(x) \geq 0$ then $x \in \mathcal{O}_P$ so $F[x] \subset \mathcal{O}_P$. Therefore $y$ is integral over $\mathcal{O}_P$, and since $\mathcal{O}_P$ is integrally closed in $K$, $\operatorname{ord}_P(y) \geq 0$. Equivalently we can say if $\operatorname{ord}_P(y) < 0$ then $\operatorname{ord}_P(x) < 0$, i.e. any pole of $y$ will be a pole of $x$. In the terminology of divisors, this means $\operatorname{supp}(\operatorname{div}_\infty(y)) \subset \operatorname{supp}(\operatorname{div}_\infty(x))$ and because the divisor of poles is effective for any $f \in K^\times$, there is some $k \in \mathbb{Z}_{>0}$ so that $\operatorname{div}_\infty(y) \leq k \operatorname{div}_\infty(x)$ and $k \operatorname{div}_\infty(x) + \operatorname{div}(y) \geq \operatorname{div}_0(y) \geq \mathbf{0}$.

Thus for any element $y$ of $R_x$, $y \in L(k \operatorname{div}_\infty(x))$ for some $k > 0$ depending on $y$. Since $K$ is degree $n$ over $F(x)$, we can find $n$ basis elements $\{y_1, \ldots, y_n\}$

in $R$ for $K$ as an $F(x)$-vector space. Since $y_i \in R$ for each $i$, we will have $y_i \in L(k_i \operatorname{div}_\infty(x))$ for some integer $k_i > 0$.

Take $k = \max\{k_1, \dots, k_n\} > 0$, so each $y_i$ will be inside $L(k \operatorname{div}_\infty(x))$. Moreover, because $x$ is transcendental over $F$, given any integer $m \geq k$, the elements $\{x^i y_j : 1 \leq j \leq n, 0 \leq i \leq m-k\}$ are linearly independent over $F$ and are inside of $L(m \operatorname{div}_\infty(x))$. Therefore $\ell(m \operatorname{div}_\infty)(x) \geq n(m-k+1)$. Recalling the notation $r(D) = \deg(D) - \ell(D)$ and noting that by Lemma 4.9, $r(E) \geq r(D)$ when $E \geq D$, we find that

$$\begin{aligned} r(m \operatorname{div}_\infty(x)) &= \deg(m \operatorname{div}_\infty(x)) - \ell(m \operatorname{div}_\infty(x)) \\ &\leq (mn) - (n(m-k+1)) = nk - n. \end{aligned}$$

By Lemma 4.6, $\{r(m \operatorname{div}_\infty(x))\}_{m \in \mathbb{Z}}$ is an increasing sequence of integers, but by the above it is bounded and thus eventually constant. Define this "eventual constant" to be $g-1$. We write $g-1$ as opposed to $g$ to ensure non-negativity of $g$ because if $m = 0$ then $m \operatorname{div}_\infty(x) = \mathbf{0}$ and $r(\mathbf{0}) = -1$.

What we now want to show is that $r(D) \leq g-1$ for all divisors $D$, not just the divisors $m \operatorname{div}_\infty(x)$ above. The method of proving this will be a simple, clever computation. For a divisor $D$, we can break up the support of $D$ into the places where $x$ has no poles, and the places where $x$ has poles (i.e. the support of $\operatorname{div}_\infty(x)$) as follows:

$$\begin{aligned} -D &= D_1 + D_2 \\ \operatorname{supp}(D_1) &\cap \operatorname{supp}(\operatorname{div}_\infty(x)) = \emptyset \\ \operatorname{supp}(D_2) &\subset \operatorname{supp}(\operatorname{div}_\infty(x)). \end{aligned}$$

First consider any place $P$ where $D_1$ is not effective. Since $x$ does not have a pole at $P$, $F[x] \subset \mathrm{O}_P$. Furthermore, $F[x] \cap \mathfrak{m}_P \neq \{0\}$ as in Example 2.11. Since $F[x] \cap \mathfrak{m}_P$ is nonzero, it is a prime ideal of $F[x]$. Take $\pi_P(x)$ to be a nonzero irreducible generating $\mathfrak{m}_P \cap F[x]$. So for some integer $m_P \geq 1$, $\operatorname{div}(\pi_P(x)^{m_P}) + D_1$ will be effective at $P$. Moreover since $F[x] \subset R_x$, $\operatorname{supp}(\operatorname{div}_\infty(\pi_P(x))) \subset \operatorname{supp}(\operatorname{div}_\infty(x))$ and so $\operatorname{supp}(\operatorname{div}_\infty(\pi_P(x))) \cap D_1 = \emptyset$. Therefore $\operatorname{div}(\pi_P(x)^{m_P}) + D_1$ will only have negative coefficients in $\operatorname{supp}(\operatorname{div}_\infty(x))$.

If we do this at every place $P$ where $D_1$ is not effective then for $f(x) := \prod_P \pi_P(x)^{m_P} \in F[x]$, $\operatorname{div}(f(x)) + D_1$ will be effective except when $x$ has a pole. Likewise since $\operatorname{supp}(D_2) \subset \operatorname{supp}(\operatorname{div}_\infty(x))$, $D_2$ is effective everywhere except where $x$ has poles. Thus $\operatorname{div}(f(x)) + D_1 + D_2 = \operatorname{div}(f(x)) - D$ will be effective outside of the support of $\operatorname{div}_\infty(x)$. If we choose a large $m \in \mathbb{Z}$, $\operatorname{div}(f(x)) - D + m \operatorname{div}_\infty(x)$ will be effective. Thus $\operatorname{div}(f(x)) + m \operatorname{div}_\infty(x) \geq D$. By Lemma 4.9,

$$r(\operatorname{div}(f(x)) + m \operatorname{div}_\infty(x)) = r(m \operatorname{div}_\infty(x)) \geq r(D).$$

If $m$ is large enough, $r(m \operatorname{div}_\infty(x)) = g-1$, so $r(D) \leq g-1$. $\square$

**Definition 4.11.** For a function field $K/F$, the *genus* of $K$ is the integer $g \geq 0$ such that

$$\begin{aligned} g &:= 1 + \max_D r(D) \\ &= 1 + \max_{m \geq 0} r(m \operatorname{div}_\infty(x)), \end{aligned}$$

where $x$ is any element of $K - F$.

**Example 4.12.** If $K = F(x)$, $\operatorname{div}_\infty(x) = (\infty)$ then $\deg(m \operatorname{div}_\infty(x)) = m$. Meanwhile $L(m \operatorname{div}_\infty(x)) = \bigoplus_{k=1}^m F x^k$ so $\ell(m \operatorname{div}_\infty(x)) = m+1$. Therefore the genus of $F(x)$ is $m - (m+1) + 1 = 0$.

**Corollary 4.13.** *For any divisor $D$ of a function field $K/F$,*

$$\ell(D) \geq \deg(D) - g + 1,$$

*where $g$ is the genus of $K$.*

*Proof.* Since $r(D) := \deg(D) - \ell(D)$ and $r(D) \leq g - 1$ for any divisor $D$ our assertion follows immediately.

□

Corollary 4.13 is the classical statement of Riemann's Theorem. We know it is sharp for $F(x)$ by the following example.

**Example 4.14.** Let $D$ be a divisor of $F(x)$ such that $\deg(D) \geq 0$. We know from Corollary 3.20 that $\ell(D) \leq \deg(D) + 1$. Since $F(x)$ has genus 0, we know from Corollary 4.13 that $\ell(D) \geq \deg(D) - g + 1 = \deg(D) + 1$. Therefore if $D$ is a divisor of $F(x)$ then either $\ell(D) = \deg(D)+1$ or $\ell(D) = 0$.

Let us show that Theorem 4.10 gives a sharp lower bound on $\ell(D)$ for any function field $K$.

**Corollary 4.15.** *There is a constant $c$ such that if $D$ is a divisor and $\deg(D) \geq c$ then $\ell(D) = \deg(D) - g + 1$.*

*Proof.* As before, let $x \in K \setminus F$, and $m$ large enough so that $r(m \operatorname{div}_\infty(x)) = g - 1$.

Let $c = m[K : F(x)] + g$. If a divisor $D$ is such that $\deg(D) \geq c$ then

$$\begin{aligned} \deg(D - m \operatorname{div}_\infty(x)) &= \deg(D) - m \deg(\operatorname{div}_\infty(x)) \\ &\geq (m[K : F(x)] + g) - m[K : F(x)] = g. \end{aligned}$$

Therefore by Corollary 4.13,

$$\begin{aligned} \ell(D - m \operatorname{div}_\infty(x)) &\geq \deg(D - m \operatorname{div}_\infty(x)) - g + 1 \\ &\geq g - g + 1 = 1, \end{aligned}$$

so $L(D - m \operatorname{div}_\infty(x)) \neq \{0\}$.Pick any nonzero $y \in L(D - m \operatorname{div}_\infty(x))$. By definition, $\operatorname{div}(y) + D - m \operatorname{div}_\infty(x) \geq \mathbf{0}$ or equivalently $\operatorname{div}(y) + D \geq m \operatorname{div}_\infty(x)$.

By Lemma 4.9,

$$r(D) = r(\operatorname{div}(y) + D) \geq r(m \operatorname{div}_\infty(x)) = g - 1.$$

But we already knew that $r(D) \leq g-1$. We must have $r(D) = \deg(D) - \ell(D) = g - 1$ and thus our statement.

□

We will see in Corollary 6.4 that the lowest value of $c$ we can use is $2g-1$.

If we combine the results of Theorems 3.20 and 4.10 we find that for a function field $K$ of genus $g$ and any divisor $D$,

$$\deg(D) + 1 - g \leq \ell(D) \leq \deg(D) + 1.$$

Since both bounds are sharp, if you know that a divisor has degree $D$, you know each of the precisely $g+1$ values that $\ell(D)$ can take on.

Beyond consequences for the space $L(D)$, Theorem 4.10 carries interesting consequences for the adeles of $K$.

**Corollary 4.16.** *For any divisor $D$ such that $\deg(D) \geq c$, where $c$ is the constant from Corollary 4.15, $A_K(D) + K = \mathbb{A}_K$*

*Proof.* By Lemma 4.6, $\dim_F(\frac{A_K(E)+K}{A_K(D)+K}) = r(E) - r(D)$ for $E \geq D$. By Corollary 4.15, $r(D) = g - 1$ for any divisor $D$ with $\deg(D) \geq c$. So if $\deg(E) \geq c$ and $\deg(D) \geq c$ then $A_K(E) + K = A_K(D) + K$. So for any divisor $D = \sum_P n_P P$ with $\deg(D) \geq c$ and any adele $(\phi_P)$, let $E = \max(D, -\operatorname{div}((\phi_P)))$. Since $E \geq D$, $\deg(E) \geq \deg(D) \geq c$ and so $A_K(E) + K = A_K(D) + K$. We also know that

$$(\phi_P) \in A_K(-\operatorname{div}((\phi_P))) \subset A_K(E) \subset A_K(E) + K = A_K(D) + K.$$

Therefore if $D$ has a high enough degree, then any arbitrary adele belongs to $A_K(D)+K$ and so $\mathbb{A}_K \subset A_K(D)+K$. However, both $A_K(D)$ and $K$(under the diagonal embedding) are subsets of the adeles, so $\mathbb{A}_K \supset A_K(D)+K$ and so our result is proven. □

## 5. Differentials and the Riemann-Roch Theorem

Now that we have a precise set of bounds for $\ell(D)$ based upon $\deg(D)$, we can find the way to precisely calculate $\ell(D)$, the Riemann-Roch Theorem. To get the full theorem, we must introduce an object called a *Weil differential.* While the definition of a differential is not difficult to state, its uses and origins are not clear based on the definition. With this in mind, we provide some motivation following [7, Chapter 6] before defining a Weil differential.

Let $X$ be a compact Riemann surface of genus $g$, $M$ the field of meromorphic functions on $X$ and $\Omega$ the space of meromorphic differential forms on $X$. Fix $\omega \in \Omega$ and a point $P \in X$. Let $t$ be a local parameter(relating to some local coordinate) at $P$. That is, pick a $t$ such that $t$ vanishes to order one at $P$. If we pick some derivation d on $X$, we could then find a power series expansion

$$\omega = \sum_{k \in \mathbb{Z}} a_k t^k \,\mathrm{d}t.$$

Since $\omega$ is meromorphic, at $P$ there is some least number $N$ where $a_k$ is nonzero. Let the order of $\omega$ at $P$ be that $N$. For a meromorphic form $\omega$, $\text{ord}_P(\omega) = 0$ for all but finitely many $P$. Therefore we can define a divisor of the points of $X$,

$$\text{div}(\omega) = \sum_P \text{ord}_P(\omega) P.$$

Likewise for a function $f$ in $M$, we could look at $\phi$ locally, finding $f = \sum_{j=-J}^{\infty} b_j t^j$. Then, by integrating over a small simple closed path around $P$, we can see that the residue of $f\omega$ at $P$ is

$$\text{Res}_P(f\omega) = c_{-1} = \sum_{i+j=-1} a_i b_j.$$

So we define a function $\omega_P : M \to \mathbb{C}$ by $f \mapsto \text{Res}_P(f\omega)$. On a Riemann surface $X$ we have the relation

$$\sum_{P \in X} \text{Res}_P(\omega) = 0,$$

commonly known as the residue theorem [5, Theorem 4.3.17], so we can say that

$$\sum_{P \in X} \omega_P(f) = 0$$

for all $f \in M$.

Let $H_P$ be the set of functions in $M$ holomorphic at $P$ and let $A_X$ be the $\mathbb{C}$-vector space inside of $\prod_P M$ such that if $\phi = (\phi_P) \in A_X$ then $\phi_P \in H_P$ for all but finitely many $P$.

For a divisor $D = \sum_P n_P P$, define $A_X(D) \subset A_X$ by

$$A_X(D) = \{\phi \in A_X : \text{ord}_P(\phi_P) + n_P \geq 0 \text{ or } \phi_P = 0 \text{ for all } P\}.$$

Let

$$\widehat{\omega}((\phi_P)) := \sum_P \omega_P(\phi_P) = \sum_P \text{Res}_P(\phi_P \omega).$$

Note that this is well-defined because if $\phi_P \in H_P$ and $\text{ord}_P(\omega) \geq 0$ then $\text{Res}_P(\phi_P\omega) = 0$ so $\widehat{\omega}((\phi_P))$ is defined by a finite sum.

Since the residue is $\mathbb{C}$-linear, so is the function $\widehat{\omega} : A_X \to \mathbb{C}$. By the definition of order, $\widehat{\omega}$ vanishes on $A_X(\text{div}(\omega))$. Since $M \hookrightarrow A_X$ by $f \mapsto (f, f, \dots)$, $\widehat{\omega}$ also vanishes on $M$ by the residue theorem. With this in mind we make the following definition of the differentials we will be using.

**Definition 5.1.** A *Weil differential* $\omega$ on a function field $K$ over an algebraically closed field $F$ is an $F$-linear map from $\mathbb{A}_K$ to $F$ which vanishes both on $K$ and on $A_K(D)$ for some divisor $D$ of $K$. The space of differentials of $K$ will be denoted $\Omega_K$ and the space of differentials which vanish on $A_K(D)$ for a fixed divisor $D$ is denoted $\Omega_K(D)$.

**Example 5.2.** Look at $F(x)$ where $x$ has a pole of order one at $\infty$ and a zero of order one at 0. The adeles of $F(x)$ will be vectors $(\phi_P)$ indexed by the points of $F$ and $\infty$, with $\phi_P \in F(x)$ for all $P$ and $\mathrm{ord}_P(\phi_P) \geq 0$ for all but finitely many $P$.

If we consider a derivation $\mathrm{d}$ of $F(x)/F$, an $F$-linear functional on $F(x)$ which obeys the Leibniz rule $\mathrm{d}(xy) = x\,\mathrm{d}y + y\,\mathrm{d}x$ and vanishes on $F$, we get a differential form $\mathrm{d}x$. The differential form $\mathrm{d}x$ gives a Weil differential $\widehat{\mathrm{d}x}$ defined via residues. For any adele $(\phi_P)$ of $F(x)$,

$$\widehat{\mathrm{d}x}(\phi_P) = \sum_P \mathrm{Res}_P(\phi_P\,\mathrm{d}x),$$

where the residue is defined as in the Riemann Surface case by picking a uniformizer and obtaining a power series expansion from the local completion of $F(x)$ at $P$.

As we saw in Example 2.11, every point $P$ corresponds to either $a \in F$ or $\infty$. So any adele $\prod_P \phi_P(x) = \prod_{a\in F} \phi_a(x) \times \phi_\infty(x)$. Since $(x-a)$ is a uniformizing parameter for $\mathrm{ord}_a$ for all $a \in F$ and $1/x$ is a uniformizing parameter for $\mathrm{ord}_\infty$,

$$\begin{aligned}\widehat{\mathrm{d}x}(\phi_P) &= \sum_{a\in F} \mathrm{Res}_{x=a}\left(\phi_a(x)\,\mathrm{d}(x-a)\right) + \mathrm{Res}_{x=\infty}\left(\phi_\infty(x)\,\mathrm{d}x\right)\\ &= \sum_{a\in F} \mathrm{Res}_{x=a}\left(\phi_a(x)\right)\,\mathrm{d}x + \mathrm{Res}_{x=\infty}\left(\phi_\infty(x)\,\mathrm{d}x\right).\end{aligned}$$

To compute the residue at $\infty$ we perform a change of coordinates, say $z = 1/x$, to get

$$\mathrm{Res}_{x=\infty}\left(f(x)\,\mathrm{d}x\right) = \mathrm{Res}_{z=0}\left(f(1/z)\,\mathrm{d}(1/z)\right) = \mathrm{Res}_{z=0}\left(-f(1/z)/z^2\,\mathrm{d}z\right),$$

because $\mathrm{d}(1/z) = -\,\mathrm{d}z/z^2$. Therefore, for any adele $(\phi_P)$ of $F(x)$, we can say

$$\widehat{\mathrm{d}x}(\phi_P) = \sum_{a\in F} \mathrm{Res}_{x=a}\left(\phi_a(x)\,\mathrm{d}x\right) + \mathrm{Res}_{z=0}\left(-\phi_\infty(1/z)/z^2\,\mathrm{d}z\right).$$

We will now use this equation to compute $\widehat{\mathrm{d}x}(\phi_P)$ for some adeles of $F(x)$: consider the adele $(f_P) = \prod_{a\in F} f_a(x) \times f_\infty(x)$ of $F(x)$ where $f_a(x) = 1$ for $a \neq \infty$ and $f_\infty(x) = 1/x$. This adele is actually holomorphic everywhere on $F(x)$. We know that if a function $f(x)$ is holomorphic at $a \in F$ then $\mathrm{Res}_a f(x)\,\mathrm{d}x = 0$, so $\widehat{\mathrm{d}x}(f_P)) = \mathrm{Res}_\infty f_\infty(x)\,\mathrm{d}x$. We know then that $\mathrm{Res}_\infty(\,\mathrm{d}x/x) = \mathrm{Res}_0(-\,\mathrm{d}z/z) = -1$ and $\mathrm{Res}_\infty(\,\mathrm{d}x/x^2) = \mathrm{Res}_0(-\,\mathrm{d}z) = 0$. Therefore $\widehat{\mathrm{d}x}(f_P) \neq 0$, but $\widehat{\mathrm{d}x}(g_P) = 0$ for the adele $(g_P) = \prod_{a\in F} g_a(x) \times g_\infty(x)$ where $g_a(x) = 1$ and $g_\infty(x) = 1/x^2$.

We see how a differential form $\omega$ gives a Weil differential $\widehat{\omega}$. It turns out that we can go the other way as well, so that a Weil differential gives a differential form $y\,\mathrm{d}(x)$([2, Section 2.5], [1, Appendix]).

We know that for all $a \in F$ and $\omega \in \Omega_K$, $a\omega$ is also an $F$-linear map from $\mathbb{A}_K$ to $F$. Also note that if $\omega$ vanishes on $A_K(D)$ then so does $a\omega$. Therefore $\Omega_K$ and $\Omega_K(D)$ are both $F$ vector spaces. In fact,

$$\Omega_K(D) = \mathrm{Hom}_F(\mathbb{A}_K/(A_K(D) + K), F) = (\mathbb{A}_K/(A_K(D) + K))^\vee,$$

because $\omega \in \Omega_K(D)$ if and only if $\omega$ comes from an $F$-linear function $\mathbb{A}_K/(A_K(D) + K) \to F$.

Using this fact, we prove a theorem which gets us another step closer to the full Riemann-Roch theorem.

**Theorem 5.3.** *For any divisor $D$, $\Omega_K(D)$ is finite dimensional over $F$ and*

$$\ell(D) = \deg(D) - g + 1 + \dim_F \Omega_K(D).$$

*Proof.* For any divisor $D$, $\Omega_K(D) = \left(\frac{\mathbb{A}_K}{A_K(D)+K}\right)^\vee = \left(\frac{A_K(E)+K}{A_K(D)+K}\right)^\vee$ for any $E \geq D$ which has large enough degree by Corollary 4.16. By Lemma 4.6 for any divisors $E \geq D$,

$$\dim_F(\frac{A_K(E) + K}{A_K(D) + K}) = r(E) - r(D) < \infty.$$

Then $\Omega_K(D)$ is finite-dimensional and $\dim_F(\Omega_K(D)) = \dim_F \left(\frac{\mathbb{A}_K}{A_K(D)+K}\right)$ via duality. In particular, since $r(E) = g - 1$ and $A_K(E) + K = \mathbb{A}_K$ for any divisor $E$ with high enough degree, we have

$$\dim_F \Omega_K(D) = g - 1 - (\deg(D) - \ell(D)).$$

Rearranging the terms in the preceding equation gives us the desired result. □

**Corollary 5.4.** *The genus $g$ is* $\dim_F \Omega_K(\mathbf{0})$.

*Proof.* By Theorem 5.3,

$$\begin{aligned} \ell(\mathbf{0}) &= \deg(\mathbf{0}) - g + 1 + \dim_F \Omega_K(\mathbf{0}) \\ 1 &= 0 - g + 1 + \dim_F \Omega_K(\mathbf{0}), \end{aligned}$$

so $\dim_F \Omega_K(\mathbf{0}) = g$. □

Our next big goal will be to find a divisor $D'$ related to $D$ where $\Omega_K(D) \cong L(D')$. It will turn out that $D'$ will be expressed in terms of $D$ and a divisor related to a nonzero differential in $\Omega_K(D)$. As such, our next step will be to determine the divisor of a differential.

**Theorem 5.5.** *For any nonzero differential $\omega$, there is a greatest divisor $D$ such that $\omega$ vanishes on $A_K(D)$. That is, $\omega$ vanishes on $A_K(E)$ if and only if $E \leq D$.*

*Proof.* Let $S_\omega$ be the set of divisors $E$ where $\omega$ vanishes on $A_K(E)$. Corollary 4.16 says that for a divisor $E$,

$$\deg(E) \geq c \Rightarrow A_K(E) + K = \mathbb{A}_K.$$

An equivalent statement is then

$$\mathbb{A}_K \neq A_K(E) + K \Rightarrow \deg(E) < c.$$

Since $\omega$ is nonzero, there is some adele $\phi$ on which $\omega$ does not vanish. Therefore if $E \in S_\omega$ then $A_K(E) + K$ is not all of the adeles, so we have a bound on the degree of divisors in $S_\omega$.

Since $\deg(\cdot)$ is integer-valued, we can fix some divisor $D$ of maximal degree in $S_\omega$. This divisor is unique. To see why, pick any other $E \in S_\omega$. By the definition of $S_\omega$, $\omega$ vanishes on both $A_K(E)$ and $A_K(D)$. Therefore it vanishes on

$$A_K(E) + A_K(D) = A_K(\max(D, E)).$$

Therefore $\max(D, E) \in S_\omega$. By definition $\max(D, E) \geq D$, so that $\deg(\max(D, E)) \geq \deg(D)$. But since $D$ has maximal degree in $S_\omega$, $\deg(D) = \deg(\max(D, E))$ and we must have $D = \max(D, E)$. Therefore $D \geq E$.

□

**Definition 5.6.** The divisor $D$ of largest degree such that $\omega$ vanishes on $A_K(D)$ is denoted $\mathrm{div}(\omega)$.

By Theorem 5.5, $\omega(\phi) = 0$ for all $\phi \in A_K(\mathrm{div}(\omega))$ and if $\omega(\phi) = 0$ for all $\phi \in A_K(E)$ then $E \leq \mathrm{div}(\omega)$.

**Example 5.7.** In Example 5.2 we saw that on $F(x)$,

$$\widehat{\mathrm{d}x}(g_P) = \widehat{\mathrm{d}x}\left(\prod_{a \in F} g_a(x) \times g_\infty(x)\right) = 0$$

when $g_a(x) = 1$ for $a \neq \infty$ and $y_\infty = 1/x^2$. Since $\mathrm{Res}_P\, f(x)\,\mathrm{d}x = 0$ for functions $f(x)$ holomorphic at $P \neq \infty$ and $\mathrm{Res}_\infty\, f(x)\,\mathrm{d}x = 0$ for functions $f(x)$ with a zero of order 2 or more at $\infty$, $\widehat{\mathrm{d}x}$ vanishes on $A_K(-2(\infty))$. Therefore by Theorem 5.5, $-2(\infty) \leq \mathrm{div}(\widehat{\mathrm{d}x})$.

We actually have equality in this case. Since $\widehat{\mathrm{d}x}(f_P) = -1$ if $f_P = 1$ for $P \neq \infty$ and $f_\infty = 1/x$, $\widehat{\mathrm{d}x}$ does not vanish on $A_K(-1(\infty))$ and so we can't improve there. Meanwhile, for each $a \in F$ if we consider the adele $(\xi(a)_P)$ where $\xi(a)_P = 1$ for all $P \neq a$ or $\infty$, $\xi(a)_P = \frac{1}{x-a}$ for $P = a$ and $\xi(a)_\infty = 1/x^2$. We find that $\widehat{\mathrm{d}x}(\xi(a)_P) = 1$ and for any $D > -2(\infty)$, either $A_K(D)$ contains $(f_P)$ above or $A_K(D)$ contains $(\xi(a)_P)$ for some $a \in F$. We then see that for any $D > -2(\infty)$, $\widehat{\mathrm{d}x}$ does not vanish on $A_K(D)$. Therefore $\mathrm{div}(\widehat{\mathrm{d}x}) = -2(\infty)$.

We already know that $\Omega_K$ is an $F$-vector space. It is also a $K$-vector space by the following rule: if $\alpha \in K$, $\omega \in \Omega_K$, and $\phi \in \mathbb{A}_K$ set

$$(\alpha\omega)(\phi) = \omega(\alpha\phi).$$

Then $\alpha\omega \in \Omega_K$.

**Lemma 5.8.** *For $\omega \in \Omega_K$ and $\alpha \in K^\times$,*

$$\operatorname{div}(\alpha\omega) = \operatorname{div}(\alpha) + \operatorname{div}(\omega).$$

*Proof.* Let $\phi \in \mathbb{A}_K$, $D = \sum_P n_P P$ and $\alpha \in K^\times$ . The key here is that

$$\begin{aligned} \alpha\phi \in A_K(D) &\iff \operatorname{ord}_P(\alpha\phi_P) + n_P \geq 0 \text{ or } \phi_P = 0 \text{ for all} P \\ &\iff \operatorname{ord}_P(\phi) + (\operatorname{ord}_P(\alpha) + n_P) \geq 0 \text{ or } \phi_P = 0 \text{ for all } P \\ &\iff \phi \in A_K(\operatorname{div}(\alpha) + D). \end{aligned}$$

So if $\omega$ vanishes on $A_K(D)$ then $\alpha\omega$ vanishes on $A_K(\operatorname{div}(\alpha) + D)$ and conversely. Therefore

$$\omega \in \Omega_K(D) \text{ if and only if } \alpha\omega \in \Omega_K(\operatorname{div}(\alpha) + D). \tag{5.1}$$

Now clearly $\omega \in \Omega_K(\operatorname{div}(\omega))$ and so $\operatorname{div}(\alpha) + \operatorname{div}(\omega) \in S_{\alpha\omega}$ where $S_{\alpha\omega}$ is defined in Theorem 5.5. Let $D = \operatorname{div}(\alpha\omega)$, so that $D \geq \operatorname{div}(\alpha) + \operatorname{div}(\omega)$.

We have $\alpha\omega \in \Omega_K(D) = \Omega_K(\operatorname{div}(\alpha) + (D - \operatorname{div}(\alpha)))$ and so by the equivalence in equation (5.1), $\omega \in \Omega_K(D - \operatorname{div}(\alpha))$. By the definition of $\operatorname{div}(\omega)$ however, $\operatorname{div}(\omega) \geq D - \operatorname{div}(\alpha)$ so $D \leq \operatorname{div}(\omega) + \operatorname{div}(\alpha)$. Since we already knew $D \geq \operatorname{div}(\omega) + \operatorname{div}(\alpha)$ this completes the proof.

□

**Lemma 5.9.** *For nonzero $\omega \in \Omega_K$ and any divisor $D$, $L(\operatorname{div}(\omega) - D)\omega \subset \Omega_K(D)$. Moreover, $L(E)\omega) \subset \Omega_K(D)$ if and only if $E \leq \operatorname{div}(\omega) - D$.*

*Proof.* Given an element $\alpha \in K^\times$,

$$\alpha \in L(\operatorname{div}(\omega) - D) \iff \operatorname{div}(\alpha) + \operatorname{div}(\omega) = \operatorname{div}(\alpha\omega) \geq D.$$

This implies that $A_K(\operatorname{div}(\alpha\omega)) \supset A_K(D)$. Since the differential $\alpha\omega$ vanishes on $A_K(\operatorname{div}(\alpha\omega))$, it also vanishes on $A_K(D)$. □

**Theorem 5.10.** *The space of Weil differentials is a one-dimensional $K$ vector space.*

*Proof.* By Lemma 5.9 we know that for any two nonzero differentials $\omega, \eta$ and any divisor $D$, $L(\operatorname{div}(\omega) - D)\omega \subset \Omega_K(D)$ and $L(\operatorname{div}(\eta) - D)\eta \subset \Omega_K(D)$. If we find a nonzero element $\rho$ in the intersection then $\rho = \alpha\omega = \beta\eta$ for nonzero $\alpha$ and $\beta$ in $K$. So $\omega$ and $\eta$ are linearly dependent over $K$.

Since $\Omega_K(D)$, $L(\operatorname{div}(\omega) - D)\omega$, $L(\operatorname{div}(\eta) - D)\eta$ are $F$-vector spaces,

$$L(\operatorname{div}(\omega) - D)\omega \subset \Omega_K(D)$$

and

$$L(\operatorname{div}(\eta) - D)\eta \subset \Omega_K(D)$$

as $F$-subspaces. Let's assume $L(\operatorname{div}(\omega) - D)\omega \cap L(\operatorname{div}(\eta) - D)\eta$ is in fact zero, and so $\Omega_K(D)$ must contain $L(\operatorname{div}(\omega) - D)\omega \oplus L(\operatorname{div}(\eta) - D)\eta$. Then we must have

$$\dim_F(\Omega_K(D)) \geq \ell(\operatorname{div}(\omega) - D) + \ell(\operatorname{div}(\eta) - D). \tag{5.2}$$

Our method will be to choose $D$ so this can't happen. Pick an integer $n \geq 1$ and a place P. If we set $D = -nP$, by Theorem 5.3,

$$\dim_F \Omega_K(-nP) = \ell(-nP) - \deg(-nP) + g - 1.$$

Because there are no nonzero functions with a zero but no poles, $\ell(-nP) = 0$. Therefore $\dim_F \Omega_K(-nP) = n + g - 1$. By Theorem 4.10 we have

$$\begin{aligned} \ell(\operatorname{div}(\omega) + nP) &\geq \deg(\operatorname{div}(\omega)) + \deg(nP) - g + 1 \\ &= \deg(\operatorname{div}(\omega)) + n - g + 1, \end{aligned}$$

and

$$\begin{aligned} \ell(\operatorname{div}(\eta) + nP) &\geq \deg(\operatorname{div}(\eta)) + \deg(nP) - g + 1 \\ &= \deg(\operatorname{div}(\eta)) + n - g + 1. \end{aligned}$$

So if

$$L(\operatorname{div}(\omega) - D)\omega \cap L(\operatorname{div}(\eta) - D)\eta = 0,$$

by Equation (5.2)we must have

$$n + g - 1 \geq 2n - 2g + 2 + \deg(\operatorname{div}(\omega)) + \deg(\operatorname{div}(\eta)).$$

But that inequality can be rearranged to

$$n \leq 3g - 3 - \deg(\operatorname{div}(\omega)) - \deg(\operatorname{div}(\eta)).$$

Since we could certainly pick a large enough $n$ to break that inequality, we have a contradiction. We have then shown that for $D = -nP$,

$$L(\operatorname{div}(\omega) - D)\omega \cap L(\operatorname{div}(\eta) - D)\eta \neq \{0\},$$

and so any two nonzero differentials are linearly dependent over $K$.

□

**Corollary 5.11.** *For any nonzero differential $\omega \in \Omega_K(D)$, $L(\operatorname{div}(\omega) - D) \cong \Omega_K(D)$ as $F$-vector spaces.*

*Proof.* By Lemma 5.9, $\alpha \mapsto \alpha\omega$ maps $L(\operatorname{div}(\omega) - D)$ into $\Omega_K(D)$ and since $K$ is a field, the map is injective. By Theorem 5.10, any nonzero differential $\eta$ can be written as $\beta\omega$ for some $\beta \in K^\times$. We want to show $\beta \in L(\operatorname{div}(\omega) - D)$ or equivalently,

$$\operatorname{div}(\beta) + \operatorname{div}(\omega) = \operatorname{div}(\beta\omega) \geq D.$$

Suppose not, i.e. $\beta \notin L(\operatorname{div}(\omega) - D)$, then $\operatorname{div}(\beta\omega) < D$ and in particular,

$$\deg(\operatorname{div}(\beta\omega)) = \deg(\operatorname{div}(\beta)) + \deg(\operatorname{div}(\omega)) = \deg(\operatorname{div}(\omega)) < \deg(D).$$

We already know however that since $\omega \in \Omega_K(D)$ that $\operatorname{div}(\omega) \geq D$ by Theorem 5.5 and so $\deg(\operatorname{div}(\omega)) \geq \deg(D)$. Therefore $\beta \in L(\operatorname{div}(\omega - D))$ and the map $\alpha \to \alpha\omega$ is an isomorphism from $L(\operatorname{div}(\omega - D)$ to $\Omega_K(D)$. □

Finally we have the tools to prove the Riemann-Roch Theorem.

**Theorem 5.12.** *For any divisor $D$ and nonzero differential $\omega$,*

$$\ell(D) = \deg(D) - g + 1 + \ell(\operatorname{div}(\omega) - D).$$

*Proof.* By Corollary 5.11, we have $\dim_F \Omega_K(D) = \ell(\mathrm{div}(\omega) - D)$ for any divisor $D$ and nonzero differential $\omega$. By Theorem 5.3 we know

$$\ell(D) = \deg(D) - g + 1 + \dim_F \Omega_K(D),$$

and by combining those two facts together we have our desired statement. □

## 6. Applications and Further Developments

The Riemann-Roch theorem has never lacked for applications, no matter how narrow your focus. Even over $\mathbb{C}$, you can find nearly endless applications (this is roughly the content of [5, Chapter 7]). Here we restate the Riemann-Roch Theorem and present a few immediate results. Let $C$ denote the divisor of any nonzero differential.

**Theorem 6.1.** *For any divisor $D$,*

$$\ell(D) = \deg(D) - g + 1 + \ell(C - D).$$

**Corollary 6.2.** *The space $L(C)$ has dimension $g$ over $F$.*

*Proof.* In Theorem 6.1 above let $D = \mathbf{0}$. Then $\ell(\mathbf{0}) = 0 - g + 1 + \ell(C)$. Since $L(\mathbf{0})$ consists of the functions without poles and thus without zeros, it is made up of just constant functions. Then $\ell(\mathbf{0}) = 1$ and the result follows. □

**Corollary 6.3.** *The degree of the divisor of any nonzero differential is $2g-2$.*

*Proof.* Use $D = C$. Then $\ell(C) = \deg(C) - g + 1 + \ell(\mathbf{0})$. Recalling the above, $g = \deg(C) - g + 2$ and so our result follows. □

Now let's revisit Corollary 4.15 and make it explicit.

**Corollary 6.4.** *For all divisors $D$ with $\deg(D) \geq 2g - 1$, $\ell(D) = \deg(D) - g + 1$.*

*Proof.* If $\deg(D) \geq 2g - 1$ then $\deg(\mathrm{div}(\omega) - D) = 2g - 2 - \deg(D) < 0$. But we showed in the proof of Theorem 4.10 that for any divisor $E$ with $\deg(E) < 0$, $\ell(E) = 0$. □

This inequality is sharp because $\deg(C) = 2g - 2$ but $\ell(C) = g$ and $g \neq 2g - 2 - g + 1 = g - 1$.

**Corollary 6.5.** *The genus of $K$ is 0 if and only if $K = F(x)$ for some transcendental $x$.*

*Proof.* We already showed in Example 4.12 that $F(x)$ is a function field of genus zero. Now consider any function field $K$ of genus zero. To apply Corollary 6.4, we need a divisor $D$ with $\deg(D) > 2(0) - 2 = -2$. Take $D = P$ for any point $P$ so by Corollary 6.4, $\ell(P) = 1 - 0 + 1 = 2$.

Clearly $F \subset L(P)$, and since $\ell(P) = 2$ there is some $x \in K - F$ which is also in $L(P)$. We see $\mathrm{div}(x) + P \geq \mathbf{0}$ so $\mathrm{ord}_Q(x) \geq 0$ for $Q \neq P$ and

$\operatorname{ord}_P(x) \geq -1$. If $\operatorname{ord}_P(x) \neq -1$ then $x$ has no poles and $x \in F$, so $\operatorname{ord}_P(x) = -1$. Since $\deg(\operatorname{div}(x) + P) = 1$ and $\operatorname{div}(x) + P$ is not supported at $P$, for some place $Q \neq P$, $\operatorname{div}(x) + P = Q$ and so $\operatorname{div}(x) = Q - P$. By Corollary 3.10, $[K : F(x)] = \deg(\operatorname{div}_0(x)) = 1$. Thus $K = F(x)$, and the only function field $K$ of genus zero is $F(x)$. □

**Remark 6.6.** Corollary 6.5 depends on the fact that there are places of degree one, which we don't necessarily have when $F$ is not algebraically closed. For an analogue in perfect fields see [4, Theorem 5.7.3].

**Corollary 6.7.** *If $K$ has genus zero then every divisor of degree zero is equal to the divisor of some $x \in K^\times$.*

*Proof.* Let $D = \sum_P n_P P$ be a divisor of degree zero. By Corollary 6.4, $\ell(D) = 1$, so we can find $x \in K^\times$ where $x \in L(D)$. By definition, $\operatorname{ord}_P(x) \geq -n_P$ for all $P$. In fact, because $\deg(D) = \deg(\operatorname{div}(x)) = 0$ and $0 = \sum_P \operatorname{ord}_P(x) \geq \sum_P -n_P = 0$ we must have $\operatorname{ord}_P(x) = -n_P$ for all $P$. Therefore $D = \operatorname{div}(x^{-1})$. □

The situation in genus one is not much more complicated.

**Theorem 6.8.** *If a function field $K/F$ has genus one then $K = F(x, y)$ with*

$$y^2 + b_1xy + b_2y = x^3 + a_1x^2 + a_2x + a_3, \tag{6.1}$$

*where $b_1, b_2, a_1, a_2, a_3 \in F$, and not all are zero.*

*Proof.* By Corollary 6.4 if $K$ has genus one and $n > 0$ then for any place $P$, $\ell(nP) = n - 1 + 1 = n$. So $L(P)$ consists only of constants. The space $L(2P)$ consists of linear combinations of constants and some $x \in K - F$ which has a pole only at $P$. Let $y$ be an element of $L(3P)$ which is linearly independent of 1 and $x$ over $F$. Then $\{1, x, y, y, x^2, xy, y^2, x^3\}$ all lie in $L(6P)$. Since $\ell(6P) = 6$ there is some linear relation

$$b_0y^2 + b_1xy + b_2y = a_0x^3 + a_1x^2 + a_2x + a_3,$$

over $F$ where $b_0, b_1, b_2, a_0, a_1, a_2, a_3$ are not all zero.

We can further say that the $a_i$'s are not all zero and the $b_i$'s are not all zero. If the $a_i$'s were all zero, then $b_0y^2+b_1xy+b_2y = 0$ and so $b_0y+b_1x+b_2 =$ in contradiction to the linear independence of $1, x$ and $y$. Likewise if the $b_i$'s were all zero, then we'd have a linear relation between $1, x, x^2$ and $x^3$.

We also have $b_0 \neq 0$ and $a_0 \neq 0$. To show this recall that $\operatorname{ord}_P(x) = -2$ and $\operatorname{ord}_P(y) = -3$. Since at least one of the $a_i$'s is nonzero, $\operatorname{ord}_P(a_0x^3 + a_1x^2+a_2x+a_3) \in 2\mathbb{Z}$ by the properties of a valuation. Therefore $\operatorname{ord}_P(b_0y^2+ b_1xy + b_2y) \in 2\mathbb{Z}$. To show $b_0 \neq 0$ suppose either $b_1$ or $b_2$ is not zero and consider $\operatorname{ord}_P(b_1xy+b_2y) = -3+\operatorname{ord}_P(b_1x+b_2)$. Since $\operatorname{ord}_P(b_1x+b_2) \in 2\mathbb{Z}$, if $b_0 = 0$ then we'd have an odd number $(\operatorname{ord}_P(b_0y^2 + b_1xy + b_2y))$ equal to an even number $(\operatorname{ord}_P(a_0x^3 + a_1x^2 + a_2x + a_3))$.

On the other hand if $a_0 = 0$ then $\operatorname{ord}_P(a_1x^2 + a_2x + a_3) \geq -4$ while $\operatorname{ord}_P(b_0y^2 + b_1xy + b_2y) = \operatorname{ord}_P(b_0y^2) = -6$, and so $a_0 \neq 0$.

Now if we replace $x$ with $b_0x/a_0$ and $y$ with $b_0y/a_0$ then we get

$$\frac{b_0^3}{a_0^2}y^2 + \frac{b_1b_0^2}{a_0^2}xy + \frac{b_2b_0}{a_0}y = \frac{b_0^3}{a_0^2}x^3 + \frac{a_1b_0^2}{a_0^2}x^2 + \frac{a_1b_0}{a_0}x + a_3.$$

If we now multiply through by $\frac{a_0^2}{b_0^3}$ and rename the constants we get the equation we are looking for.

Since we have our equation, we know $K \supset F(x,y)$ with $x$ and $y$ behaving as we wish. Since $F(x,y) \neq F(x)$, $[F(x,y) : F(x)] \geq 2$. However since $x \in L(2P) - L(P)$, $\deg(\mathrm{div}_\infty(x)) = [K : F(x)] = 2$. Therefore $[K : F(x,y)] = 1$ and we have proved our assertion. $\square$

We can also go in reverse, so that for a function field $F(x,y)$ satisfying (6.1) and some additional requirements on $\{b_1, b_2, a_1, a_2, a_3\}$, we will find that $F(x,y)$ has genus 1. For a more detailed discussion, see [8, Prop 3.1.4 and Prop 3.3.1]. We also have an analogue of Corollary 6.7 in the genus one case [8, Corollary 3.3.5].

The reverse direction of Theorem 6.8 makes use of a famous formula called the *Riemann-Hurwitz formula* which can also be proven using Riemann-Roch. To state the formula, let us consider a finite extension of function fields $L/K$. Let $\mathfrak{P}$ refer to a place of $L$ and recall that since $\mathfrak{P}$ is a discrete valuation on $L^\times$ it restricts to a homomorphism (possibly not onto) from $K^\times$ to $\mathbb{Z}$. Let $e_\mathfrak{P} = [\mathrm{ord}_\mathfrak{P}(L^\times) : \mathrm{ord}_\mathfrak{P}(K^\times)] = [\mathbb{Z} : \mathrm{ord}_\mathfrak{P}(K^\times)]$. If $e_\mathfrak{P} = 1$ then because $\mathfrak{P}$ is trivial on $F^\times$, $\mathfrak{P}$ is also a place $P$ of $K^\times$. For this reason, we call $e_\mathfrak{P}$ the ramification index of $\mathfrak{P}$ over $K$. We use this information to tell us about the genus of $L$.

**Corollary 6.9.** *Let $K$ be a function field over an algebraically closed field $F$ of characteristic zero and let $L/K$ be a finite extension of $K$. Then*

$$2g_L - 2 = (2g_K - 2)[L : K] + \sum_{\mathfrak{P}} (e_\mathfrak{P} - 1),$$

*where the sum ranges over all places $\mathfrak{P}$ of $L$.*

*Proof.* See [7, Theorem 7.16], not just for a proof but for a description of what happens when the characteristic of $F$ is non-zero. $\square$

Our approach to the Riemman-Roch theorem in this paper can be made to work over a general perfect field as opposed to an algebraically closed field, and that's the approach taken in [1], [4], [6] and [7]. There is another method of proof using *Serre duality.* With that method one can prove Riemann-Roch over an arbitrary ground field. For a proof in this fashion see [3, pg. 316].

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