TOPICS IN MODERN GEOMETRY
TOPOLOGY
HOMEWORK SHEET 1 SOLUTIONS

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Exercise 1. Firstly, we must check the axioms given in the lecture notes to show that this collection τ of subsets defines a topology. By construction we have that $\emptyset \in \tau$, and on the other hand $X \backslash X = \emptyset$ is countable so $X \in \tau$. Let $A, B \in \tau$, then $X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B)$ is a (countable) union of countable sets, hence by induction τ is closed under finite intersections. Finally, let I be a (possibly uncountable) set and let $\forall i \in I, U_i \in \tau$ be an indexed family of open sets. We have, for any $j \in I$, $X \backslash \cup_{i \in I} U_i = \cap_{i \in I}(X \backslash U_i) \subseteq X \backslash U_j$, that is the intersection is a subset of a countable set, hence countable, and τ is closed under arbitrary unions.
Now let X be a finite set, let τ be the co-countable topology and let δ be the discrete topology. As every subset is open with respect to the discrete topology, we have $\tau \subseteq \delta$. Conversely, any the complement of any subset of a countable set is also a subset and is thus countable, so we see that $\delta \subseteq \tau$. We conclude that $\tau = \delta$.

Exercise 2. I remark that continuity in the metric sense is a special case of topological continuity, so you may have already seen a valid $\varepsilon - \delta$ proof that polynomial maps $\mathbb{R}^m \to \mathbb{R}^n$ are continuous. The question here is the first steps towards a topological proof of this fact, given purely in terms of open sets. The essential idea is that a polynomial map is simply a composition of multiplications and additions. For example, we note that constant maps are continuous, as the pre-image of any open set in $\mathbb{R}$ is either $\emptyset$ or $\mathbb{R}$, both of which are open by definition. If multiplications and additions are continuous, it follows that linear maps are continuous as each linear polynomial is given by a multiplication by a constant and an addition of a constant. Similarly quadratic maps are obtained by multiplying by a linear expression and adding a constant. One could turn this into a formal induction if required, but it simply suffices to say that a real polynomial is given by a composition of continuous maps (multiplications and additions in $\mathbb{R}$) and is thus continuous. Note that this argument can be extended to polynomials in several variables.

Exercise 3. Every subset is open with respect to the discrete topology. Given a set of size $N+1$, there are 2^{N+1} subsets. In the case of the set $\{0, 1\}$, there are 4 subsets, namely $\emptyset, \{0, 1\}, \{0\}, \{1\}$.

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