TOPICS IN MODERN GEOMETRY
TOPOLOGY
HOMEWORK SHEET 2 SOLUTIONS

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Solution 1. Assume X is a metric space and let $x, y \in X$ be distinct points. Consider the open balls $B_x := B_{\frac{d(x,y)}{2}}(x) \ni x$ and $B_y := B_{\frac{d(x,y)}{2}}(y) \ni y$. One shows that these balls are disjoint by contradiction as follows. If $z \in B_x \cap B_y$, then $d(z,x), d(z,y) < \frac{d(x,y)}{2}$, so $d(z,x) + d(z,y) < d(x,y)$, which contradicts the triangle inequality.

Solution 2. Recall that the circle S^1 is the boundary of the disc. It is not the disc itself! Note that a point in $S^1 \times [a,b]$ can be written

$$(\cos\theta, \sin\theta, x),$$

with $(\cos\theta, \sin\theta) \in S^1$, $x \in [a,b]$. It should be clear that this parametrises a cylinder C in $\mathbb{R}^3$. The real purpose of this question is to show that the product topology on $S^1 \times [a,b]$ agrees with the subspace topology on $C \subset \mathbb{R}^3$.
To do this, let U be a basis open set of C with respect to the subspace topology from $\mathbb{R}^3$. Recall that one can take open cubes as a basis of $\mathbb{R}^3$. It follows that, U is the intersection of an open cube in $\mathbb{R}^3$ with C, which geometrically looks like an open square on C, that is a square on the curved surface of C. A basis open set V in $S^1 \times [a,b]$ with respect to the product topology is the cartesian product of an open arc in the circle with an open interval in $[a,b]$, such a thing is exactly what we have just described! So the bases agree, and hence the topologies agree.
The annulus is homeomorphic to the cylinder by projection onto the plane. One can even write down such a map explicitly:

$$p : C \to \mathbb{R}^2$$
$$p(\cos\theta, \sin\theta, x) = (x\cos\theta, x\sin\theta).$$

This map is a homeomorphism onto its image, the annulus, as it is a composition of polynomial and trigonometric maps, both of which are continuous.
Next we must show that the torus is homeomorphic to the Cartesian product $S^1 \times S^1$ and the quotient of the unit square $[0,1]^2$ constructed by identifying opposite sides. The cheater's way to do this (which is perfectly valid) is to say that all three of these spaces are compact, orientable 2-dimensional manifolds of genus 1, and hence they must be homeomorphic by the classification of surfaces. Alternatively one can show that there are inclusions of topological bases similarly to above. For example, a basis open set in the quotient topology is a subset who's pre-image with respect the the quotient map is an open set in the unit square. A basis open set with respect to the subspace topology from $\mathbb{R}^3$, which is the intersection of an open ball with the torus, is clearly such a set.

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Solution 3. One must show that multiplication and inversion of pairs is continuous with respect to the product topology. This follows from the fact that if $f : X \to Y$ and $g : X \to Z$ are continuous, then $(f, g) : X \to Y \times Z$ is continuous. This claim follows from the fact that $(f, g)^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V)$ which implies the inverse image of a basis open set is an intersection of open sets, hence open.