TOPICS IN MODERN GEOMETRY
TOPOLOGY
HOMEWORK SHEET 3 SOLUTIONS

THOMAS OLIVER
UNIVERSITY OF BRISTOL

Exercise 1. Let X be a finite set with the discrete topology. Take an open cover $\cup_{\alpha \in A} U_\alpha$ of X. Then, for all $x \in X$, there is some $\alpha_x \in A$ such that $x \in U_{\alpha_x}$. This means that $X = \cup_{x \in X} U_{\alpha_x}$ is a finite subcover and X is compact. For the second part, consider the subset $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$ with the subspace topology from $\mathbb{R}$, which is closed and bounded, hence compact. This space contains the infinite discrete subspace $\{\frac{1}{n} : n \in \mathbb{N}\}$, which is not closed, as it does not contain the limit point 0.

Solution 2. Let $f : X \to Y$ be a continuous map, this induces a surjective continuous map $f : X \to f(X)$. For the first part, assume X is connected and take an open cover $\cup U_\alpha$ of $f(X)$. The union $\cup_\alpha f^{-1}(U_\alpha)$ is a cover of X and hence has a finite subcover $\{f^{-1}(U_{\alpha_n}) : n = 1, \ldots, N\}$. The sets U_{α_n} then cover $f(X)$. For the second part, assume X is connected. If Y is not connected, then Y may be written as a disjoint union of non-empty open sets $Y = A \cup B$. We can then write X as the disjoint union $X = f^{-1}(A) \cup f^{-1}(B)$, in which both pre-images are non-empty as f is surjective. As f is continuous, both $f^{-1}(A)$ and $f^{-1}(B)$ are open. This is a contradiction to the connectedness of X.

Exercise 3. First we will present an argument based on compactness. We note that S^1 is a closed and bounded subset of $\mathbb{R}^2$ and is thus compact, on the other hand $\mathbb{R}$ is not bounded and is not compact. If there was a homeomorphism from S^1 to $\mathbb{R}$, then $\mathbb{R}$ would be the continuous image of a compact set and therefore compact. This is a contradiction. Now we will give a connectedness argument. For any $x \in \mathbb{R}$, $\mathbb{R} - \{x\}$ is not connected, wheras for any $y \in S^1$, $S^1 - \{y\}$ is connected. If there was a homeomorphism from $S^1 \to \mathbb{R}$, then there is also a homeomorphism $S^1 - \{y\} \to \mathbb{R} - \{f(y)\}$ and so $\mathbb{R} - \{f(y)\}$ would be the continuous image of a connected set, hence connected. This is again a contradiction.

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