TOPICS IN MODERN GEOMETRY
TOPOLOGY PROBLEMS CLASS 1 SOLUTIONS

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Exercise 1. Consider the following family of subsets of $\mathbb{R}$

$$\forall a \in \mathbb{R}, I_a := (a, \infty);$$

$$I_\infty := \emptyset;$$

$$I_{-\infty} := \mathbb{R}.$$

Show that the collection of subsets $\{I_x : x \in \mathbb{R} \cup \{\pm\infty\}\}$ defines a topology on $\mathbb{R}$. Given $A \subseteq \mathbb{R}$, what is the closure of A?

Solution. To solve the first part of this exercise, we simply check the axioms given in lecture one. Let τ denote the collection given in the question. Clearly $\emptyset, \mathbb{R} \in \tau$. Next, let $\{I^i : i \in I\}$ be an indexed family of sets in τ, the union

$$\cup_{i\in I} I^i = \begin{cases} \mathbb{R}, & \exists i \text{ st } I^i = \mathbb{R}; \\ I_{\inf(a:\exists i \text{ st } I^i = I_a)}, & \text{otherwise.} \end{cases}$$

So, τ is closed under arbitrary unions. Finally, given a finite collection $I^1, I^2, I^3 \ldots, I^n$ of sets in τ, we have

$$\cap_{i\in\mathbb{N}} I^i = \begin{cases} \emptyset, & \exists i \text{ st } I^i = \emptyset \\ I_{\sup(a:\exists i \text{ st } I^i = I_a)}, & \text{otherwise.} \end{cases}$$

We thus see that τ is closed under finite intersections. The closed sets with respect to τ are those whose complement is open, that is, those of the form $\emptyset, \mathbb{R}, (-\infty, a]$. By definition, the closure of a set $A \subset \mathbb{R}$ is the smallest closed set with respect to τ containing A, which is $(-\infty, \sup A]$.

Exercise 2. For $a, b \in \mathbb{Z}$, $b > 0$, set

$$N_{a,b} := \{a + nb : n \in \mathbb{Z}\} \subset \mathbb{Z}.$$

Call a set $A \subseteq \mathbb{Z}$ *open* if A is either empty, or if to every $a \in A$ there exists some $b > 0$ with $N_{a,b} \subseteq A$. Show that:

(1) The above construction defines a topology on $\mathbb{Z}$;
(2) Any non-empty open set is infinite;
(3) Any set of the form $N_{a,b}$ is closed;
(4) $\mathbb{Z}\backslash\{-1, 1\} = \cup_{p \text{ prime}} N_{0,p}$.

Conclude that there are infinitely many prime numbers.

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Solution. (1) By definition, the empty set is an open set with respect to this topology. $\mathbb{Z}$ itself is open as it contains *all* $N_{a,b}$. Let open sets U_i contain N_{a_i,b_i}, then the union of U_i contains each N_{a_i,b_i}, and so is open. If U_1 and U_2 are open sets and $a \in U_1 \cap U_2$ with $N_{a,b_1} \subseteq U_1$ and $N_{a,b_2} \subseteq U_2$, then $a \in N_{a,b_1 b_2 \subseteq U_1 \cap U_2}$. This implies that we have a topology on $\mathbb{Z}$.

(2) A non-empty open set necessarily contains an infinite set $N_{a,b}$ and so is infinite.

(3) An open set $N_{a,b}$ can be written as the complement of a union of open sets

$$N_{a,b} = \mathbb{Z} \setminus \cup_{i=1}^{b-1} N_{a+i,b}.$$

and hence is closed.

(4) Any number $n \neq \pm 1$ has a prime divisor, which we shall call p. We have that $n \in N_{0,p}$. Consequently

$$\mathbb{Z} \setminus \{\pm 1\} = \cup_{p\text{: prime}} N_{0,p}.$$

We make the final conclusion by contradiction. Indeed, if the set of prime numbers was finite, then $\cup_p N_{0,p}$ would be a finite union of closed sets, which must be closed. Consequently, $\{\pm 1\}$ would be open, but we know that all open sets are infinite.

Exercise 3. Show that the following pairs of spaces are homeomorphic.

(1) The annulus $\{z \in \mathbb{C} : a < |z| < b\}$, for real numbers $a > b > 0$, and the bounded cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = c > 0, 0 < |z| < d\}$;
(2) An American doughnut and a mug;
(3) The letter A and the letter R (viewed as a union of curve segments in $\mathbb{R}^2$).

On the other hand, show that the letter A cannot be isomorphic to the letter B.

Solution. We will give "descriptive" homeomorphisms, leaving explicit homeomorphisms to the dedicated student.

(1) The annulus is a projection of the cylinder onto the plane. Such maps are homeomorphisms.
(2) This is the classic topology example. The key point is that each space has only one "hole". The homeomorphism comes from squashing and stretching one space into the other without making any holes or tears, as demonstrated in a lot of videos online (look it up!). This is actually part of the classification of compact orientable surfaces. Up to homeomorphism, all such surfaces are characterised by their *genus*, that is, the number of "holes".
(3) This is similar to the above. With that in mind the second part of the question is clear. The British doughnut and cylinder are both compact orientable surfaces of genus 0, and thus homeomorphic to each other and the sphere.

The idea here is that removing any two points from the letter A creates a disjoint union of pieces. On the other hand, no two points can be removed from the letter B with the same result.

Exercise 4. Show that the (compact, orientable) surface of genus 2 can be realised as a quotient of an octagon. What about a compact orientable surface of arbitrary genus?

Solution. An appropriate quotient of the octagon was drawn in class. A surface of arbitrary genus g can be realised as a quotient of a $4g-$gon.