TOPICS IN MODERN GEOMETRY
TOPOLOGY PROBLEMS CLASS 1 EXERCISES

THOMAS OLIVER
UNIVERSITY OF BRISTOL

In this problems class we will work through some exam-type questions based on the material from lectures 1 - 4.

Exercise 1. Consider the following family of subsets of $\mathbb{R}$

$$\forall a \in \mathbb{R}, I_a := (a, \infty);$$
$$I_\infty := \emptyset;$$
$$I_{-\infty} := \mathbb{R}.$$

Show that the collection of subsets $\{I_x : x \in \mathbb{R} \cup \{\pm\infty\}\}$ defines a topology on $\mathbb{R}$. Given $A \subseteq \mathbb{R}$, what is the closure of A?

Exercise 2. For $a, b \in \mathbb{Z}$, $b > 0$, set

$$N_{a,b} := \{a + nb : n \in \mathbb{Z}\} \subset \mathbb{Z}.$$

Call a set $A \subseteq \mathbb{Z}$ *open* if A is either empty, or if to every $a \in A$ there exists some $b > 0$ with $N_{a,b} \subseteq A$. Show that:

(1) The above construction defines a topology on $\mathbb{Z}$;
(2) Any non-empty open set is infinite;
(3) Any set of the form $N_{a,b}$ is closed;
(4) $\mathbb{Z} \backslash \{-1, 1\} = \cup_{p \text{ prime}} N_{0,p}$.

Conclude that there are infinitely many prime numbers.

Exercise 3. Show that the following pairs of spaces are homeomorphic.

(1) The annulus $\{z \in \mathbb{C} : a < |z| < b\}$, for real numbers $a > b > 0$, and the bounded cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = c > 0, 0 < |z| < d\}$;
(2) An American doughnut and a cup, a British doughnut and a saucer;
(3) The letter A and the letter R (viewed as a union of curve segments in $\mathbb{R}^2$).

On the other hand, show that the letter A cannot be isomorphic to the letter B.

Exercise 4. Show that the (compact, orientable) surface of genus 2 can be realised as a quotient of an octagon. What about a compact orientable surface of arbitrary genus?

Date: September - October, 2015.