"TOPICS IN MODERN GEOMETRY"
TOPOLOGY

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Introduction

This short course on topology is the first part of the module "Topics in Modern Geometry". As such, an important purpose of these lectures is to develop the pre-requisites for the second part, in which you will study hyperbolic geometry. The student should not deduce from this that topology is simply a tool for hyperbolic geometry. Indeed, topology is a vast subject, the major theorems and vocabulary of which arise in a wide range of mathematical disciplines. Nor is topology short of real word applications, making an appearance in the fields of condensed matter physics, data analysis, electronics, and even robotics.

A 9 hour course could never cover everything and our intention here is simply to introduce and explore a few key areas. First and foremost amongst which we have the question what exactly is topology? In short, it is a language with which we can speak rigorously of intuitive ideas concerning *continuity*. Indeed, it is in a sense the *minimal* such language. There is no need for our old friends ε and δ to discuss continuous maps between *topological spaces*, only a notion of *open set*. Topology can be thought of as qualitative geometry, as opposed to, say, analytic or differential geometry, in which measurements such as curvature are the key invariants. Topological invariants are those which remain after continuous deformation - one may stretch and squash, but not tear.

At first, our topological spaces will be plain old sets, but before long we will introduce algebraic and Euclidean structures to provide some context. In the algebraic case we will consider *topological groups* in which the group operation is a continuous map, wheras in the Euclidean setting the key objects are *manifolds*, of which we will give an *intrinsic definition*. We will then explore the interactions of topological and algebraic structure, placing particular focus on *discrete groups*. The simplest examples of discrete groups are finite groups, and generalising finiteness we introduce the notion of *compactness*. All discrete spaces, finite or not, are *totally disconnected*. We will quantify this statement by providing a formal definition of connectedness. Building on that, we introduce homotopy and define the *fundamental group*, which is basic invariant of a topological space. This notion will be used in the hyperbolic geometry course and is developed further in the Algebraic Topology module.

To summarise, the key topics of this course are as follows:

- General topology;
- Manifolds;
- Topological groups;
- Discrete groups;
- Compactness and connectedness.

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1. Topological Spaces

Of course, we begin with the following central definition.

Definition 1.1. A *topological space* is a pair (X, τ), in which X is a set and τ is a collection of subsets of X such that the following conditions are satisfied:

(1) $\emptyset, X \in \tau$;
(2) If A is a set[1] and $\{U_\alpha : \alpha \in A\} \subset \tau$, then $\cup_{\alpha \in A} U_\alpha \in \tau$;
(3) If, for $n \in \mathbb{N}$, $U_1, \ldots, U_n \in \tau$, then $U_1 \cap \cdots \cap U_n \in \tau$.

In the above definition, τ is called the *topology* on X and the sets in τ are the *open sets*.

The first condition says that, in any topological space, the empty set and the space itself are open subsets. The second says that the topology is closed under *arbitrary* unions, whereas the third says that the topology is closed under *finite* intersections. The terminology "open" is supposed to reflect eponymous concepts you will have seen in earlier courses, such as the notion of open interval in $\mathbb{R}$ or open ball in $\mathbb{C}$, both of which are open sets in *metric space*.

Example 1.2. Let (X, d) be a metric space and say that a set $U \subset X$ is *open* if, for any $x \in U$, there exists $\varepsilon > 0$ such that the *open ball* $B_\varepsilon(x) := \{y \in X : d(x, y) < \varepsilon\}$ is contained in U. It is left as an exercise for the student to check that this defines a topology on X. In particular, recall that the set $\mathbb{R}^n$ forms a metric space with respect to the Euclidean distance. The associated topology is called the *Euclidean topology*. In the case that $X = \mathbb{R}$, the Euclidean topology τ can be specified explicitly as

$$\tau = \{U \subseteq \mathbb{R} : \forall x \in U, \exists (a, b) \subseteq U \text{ st } x \in (a, b)\}.$$

One may moreover write down similar results fo $\mathbb{R}^n$ when $n > 1$. Notice that an infinite intersection of open sets need not be open. In $\mathbb{R}$, for example, the intersection of open intervals $\cap_{n \in \mathbb{N}}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$, which is not open. We remark that $\mathbb{C}^n$ with the usual norm can be identified with $\mathbb{R}^{2n}$ with the Euclidean norm - this means that $\mathbb{C}^n$ also forms a topological space.

So, in topology we have open sets, but what about closed sets?

Definition 1.3. Let (X, τ) be a topological space. A set $A \subset X$ is closed if its complement $F = X \backslash A$ is open.

For example, the closed interval $[a, b] \subset \mathbb{R}$ has complement

$$\mathbb{R} - [a, b] = (-\infty, a) \cup (b, \infty),$$

which is a union of open intervals. Therefore $[a, b]$ is a closed subset of $\mathbb{R}$ with respect to the Euclidean topology. We must be careful - in spite of the suggestive terminology, and the apparent agreement with our intuition, "open" and "closed" are not antonyms! For example, one may write down an examples of subsets which are both open and closed, or neither open nor closed. Sets that are both open and closed are sometimes called *clopen*[2]. The definition of topology is given in terms of a family of open sets satisfying certain axioms. In fact, one

[1] possibly uncountable!

[2] If you think that sounds like a joke, I encourage you to search YouTube for "Hilter learns topology".

could formulate an *equivalent* definition in terms of closed sets via De Morgan's laws, which we recall say that if A and B are sets, then

$$(A \cup B)^c = A^c \cap B^c,$$
$$(A \cap B)^c = A^c \cup B^c,$$

where S^c denotes the complement of a set S. In is a worthwhile exercise for the reader to write down three axioms defining a topological space in terms of closed sets.

Above we have seen that metric spaces are topological spaces. In fact, many of the familiar notions from metric spcaes carry over to the more general context of topological spaces. For example:

Definition 1.4. Let (X, τ) be a topological space

(1) Given a point $x \in X$, a *neighbourhood* of x is any subset of X containing an open set which contains x;
(2) Given a subset $A \subset X$ the *closure* $\bar{A}$ of A is the smallest closed set containing A, that is, the intersection of all closed sets containing A. Equivalently, $x \in \bar{A}$ if and only if for any neighbourhood N of x, $N \cap A \neq \emptyset$;
(3) Given a subset $A \subset X$, the *interior* $\mathrm{int}(A)$ of A is the union of all open sets contained in E.

Many of the standard results follow immediately.

Proposition 1.5. *Let X be a topological space and let $A \subset X$ be a subset. The interior of A is open and the closure of A is closed. Moreover, A is open iff A included in its interior, and A is closed iff it includes its closure.*

Proof. This is an exercise for the student. □

We have above defined a topology in $\mathbb{R}$ by taking advantage of its metric structure, though we didn't have to. Indeed, a set can have more than one topology, some examples of which are given below. That said, some topologies may be considered more "natural" than others. For example, one often seeks a topology which interacts well with other structures on X. For example, the topology given above seems to be the natural candidate on a metric space. In later lectures we will be interested in topologies that interact well with group structure on X.

Example 1.6. Throughout, let X be a set. In each case the student might wish to check that τ satisfies the axioms.

(1) The *trivial topology* on X is given by $\tau = \{\emptyset, X\}$.
(2) At the other extreme, one has the *discrete topology* in which every subset of X is open, that is $\tau = 2^X$, the power set of X.
(3) The *cofinite topology* has open sets $\emptyset$ and any $U \subset X$ such that $X \backslash U$ is finite.

Having many different topologies on a set X, one may seek to compare them. One way to do that is via inclusion. If τ_1 and τ_2 are topologies on a set X such that $\tau_1 \subset \tau_2$, then we say that τ_1 is *coarser* or *weaker* than τ_2 and that τ_2 is *finer* or *stronger* than τ_1. Notice that the discrete topology is the finest topology and the trivial topology is the coarsest.

Example 1.7. Let I be an indexing set and let τ_i, $i \in I$, be a collection of topologies on a set X. The intersection $\tau := \cap_{i \in I} \tau_i$ defines a topology on X which is coarser than each τ_i.

2. Continuity, Subspaces and Density

In this section we will study maps which preserve topological structure. Such maps are called *continuous*, the terminology reflecting the fact that this notion generalises the familiar notion on metric spaces.

Definition 2.1. Given topological spaces X and Y, a map $f : X \to Y$ is said to be continuous if for every open subset $U \subset Y$, the inverse image $f^{-1}(U) := \{x \in X : f(x) \in U\}$ is open in X.

This definition is rather concise. The purpose of the following exercise is to check that it encapsulates what we expect in the context of metric spaces.

Exercise 2.2. Let (X, d_X) and (Y, d_Y) be metric spaces, which we can think of as a topological space as in the previous lecture. Recall that a map $f : X \to Y$ is continuous if

$$\forall x \in X, \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t } \forall x' \in X \ (d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon).$$

Prove that a continuous map of metric spaces is a continuous map in the topological sense.

In particular this means that many familiar functions $\mathbb{R}^n \to \mathbb{R}^m$ are continuous, such as polynomials[3], exponentials, trigonometric functions, etc. Contrasting that given above, we make the following definition, which looks similar but is rather different in practice - so be careful!

Definition 2.3. A map $f : X \to Y$ between topological spaces X and Y is called *open* if, for any open set $U \subset Y$, the image $f(U) := \{f(x) : x \in U\}$ is open in Y.

The composition of two continuous maps is again continuous, as may easily be checked. As in all branches of mathematics, we classify topological spaces up to some specified form of equivalence. For example, one might think of two groups as "the same" if they are isomorphic, or consider two metric spaces to be "the same" if they are isometric. In topology, the analogous statement is that two spaces are "the same" if they are *homeomorphic*. This is the subject of the following definition.

Definition 2.4. Given two topological spaces X and Y, a homeomorphism $f : X \to Y$ is a bijective map which is both open and continuous.

It can be useful to note that a map f between topological spaces is a homeomorphism if and only if it is bijective and both f and f^{-1} are continuous. It can be rather difficult to say whether or not two topological spaces are homeomorphic, let alone give an explicit homeomorphism. It is for this reason one introduces *topological invariants*, certain characteristics of topological spaces which are preserved by homeomorphism. It follows that if such an invariant is different for two topological spaces, they cannot be homeomorphic. At this point a word of warning is perhaps in order, in the case that various topological invariants of two spaces agree, it need not follow that these spaces are homeomorphic.

Example 2.5. A torus can not be homeomorphic to a sphere. The invariant here is *connectedness*, which we will discuss later in this course. The idea is that removing a circle from the surface of a sphere always divides the sphere into two disjoint open subsets. On the other hand, the analogous statement for the torus is not true.

[3] A different, purely topological, proof of the continuity of polynomials will be given in the exercises.

Connectedness aside, the first difficulty we face with this argument is that we speak of "open subsets" of the sphere and the torus, and yet, we have not so far defined a topology on these spaces. One intuitive way to do this is to emded these spaces into the Euclidean space $\mathbb{R}^3$ and "induce" a topology from there. We are lead to the idea of studying subsets of topological spaces which admit topological structure in their own right. This may be compared to subgroups of a group, subspaces of a vector space, etc.

Definition 2.6. Let Y be a subset of a topological space X. The subspace topology on Y is the coarsest topology such that the map $i : Y \to X$ is continuous. That is, the open sets of Y are given by intersections $Y \cap U$ in which U is an open subset of X[4].

Example 2.7. The complex plane $\mathbb{C}$, with respect to the Euclidean metric, is homeomorphic to the punctured sphere $S^2 - \{\text{pt}\}$, which has the topology induced from the Euclidean topology on $\mathbb{R}^3$. A homeomorphism $S^2 - \{\text{pt}\} \to \mathbb{C}$ is given by *stereographic projection*.

Example 2.8. A topological space may well be homeomorphic to a proper subset. For real numbers $a < b$, the open interval (a, b) is homeomorphic to $\mathbb{R}$. You should be able to construct a homeomorphism, for example, via the function arctangent function.

Whilst not always homeomorphic to their ambient space, subspaces may still be *dense*. This means that, given any element of a topological space, there exists an element of the subspace arbitrarily "close" to it. Formally we make the following definition.

Definition 2.9. A subset Y of a topological space X is *dense* if, for any non-empty open $U \subset X$, we have $U \cap Y \neq \emptyset$. Equivalently, Y is dense in X if and only if the closure of Y is equal to X.

Exercise 2.10. Endow $\mathbb{R}$ with the Euclidean topology. Show that the subspace topology on the set of natural numbers $\mathbb{N}$ and the set of integers $\mathbb{Z}$ is the discrete topology, but the subspace topology on $\mathbb{Q}$ is not discrete. Show that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.

One sometimes refers to the reals as a *completion* of $\mathbb{Q}$. This may seem like a somewhat unique situation, but that is far from the case. In fact there are infinitely many completions of $\mathbb{Q}$, corresponding to infinitely many choices of metric. This is the topic of one of the fourth year project options. Generalising the notion of density, we have

Definition 2.11. A subset S of a topological space is *somewhere dense* if there exists a non-empty open set U such that $S \cap U$ is dense in the subspace topology of U. A subset S of a topological space X is *nowhere dense* if it is not somewhere dense.

Exercise 2.12. Show that a subset of a topological space X is nowhere dense if and only if the closure of S has empty interior.

Notice that the order in which we take interior and closure is crucial in the above statemet, as evidenced by the following exercise.

Exercise 2.13. Give an example of a subset S of a topological space X which is somewhere dense such that its interior has empty closure.

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[4]The reader might like to confirm that the two versions given in the definition above are indeed equivalent.

3. Topological Bases and Separability

Having to specify every open subset in a topological space can be somewhat tiresome. In the case of the real line $\mathbb{R}$, the key open subsets are the open intervals - all other subsets are generated from these through a process of taking unions and intersections. This is the idea of a *basis* of open sets, from which all other open sets are built. We begin by defining the closely related notion of *generated topology* associated to a given collection of subsets, which is somewhat comparable to the span of a given set of vectors in a vector space. In this analogy, a basis is a spanning set, though we note that there is no immediately evident "independence" condition.

Definition 3.1. Let σ denote a collection of subsets of a set X. The topology *generated* by σ is the intersection of all topologies on X which contain σ.

That is, the topology generated by σ is the coarsest topology on X for which each set $U \in \sigma$ is open. In the following exercise you will write down what the generated topology actually looks like, at least under a mild assumption.

Exercise 3.2. Suppose that $\sigma := \{U_\alpha \subset X : \alpha \in A\}$ is such that $X \subset \cup_\alpha U_\alpha$. In ths case we say that σ *covers* X. Show that, if σ covers X then the open sets in the topology generated by σ coincide with those sets which can be written as the union of finite intersections of elements in σ.

Definition 3.3. Let (X, τ) be a topological space. A subset $\sigma \subset \tau$ forms a basis for τ if every element of τ can be written as a union of elements of σ.

The reader might like to convince themselves that if σ is a base for τ then the topology generated by σ is equal to τ. On the other hand, one might ask when a collection of subsets is a base for the topology it generates.

Exercise 3.4. Show that if σ covers X and the intersection of any two elements of σ equals a union of elements of σ then σ is a base for the topology it generates.

Exercise 3.5. Let (X, τ) be a topological space and let $\sigma \subset \tau$ cover X. Write down a basis for the topology generated by σ.

For example, the open balls form a base for the metric topology on a metric space. In particular, as we have already mentioned, the open intervals form a basis for $\mathbb{R}$. In fact, we may refine the open interval basis for $\mathbb{R}$ to a *countable* basis, taking advantage of the density of $\mathbb{Q}$. It is down to the reader to provide details for this claim.

Definition 3.6. A topological space X is said to be *second countable* if it has a countable base.

For example a subspace of $\mathbb{R}^n$ is second countable with respect to the Euclidean topology. On the other hand $\mathbb{R}^2$ is not second countable with respect to the *order topology*. Of course, the terminology makes one wonder what a first countable topological space is. To that end:

Definition 3.7. A topological space X is *first countable* if every $x \in X$ has a countable family of open neighbourhoods $(B_i)_{i \in \mathbb{N}}$ such that any neighbourhood of x contains some B_i.

What remains is to pin down what exactly the relationship between these two is. That is the subject of the following exercise.

Exercise 3.8. Show that a topological space X is first countable if and only if

$$\forall x \in X, \exists \text{ open sets } \{U_i\} \text{ such that } \forall \text{ open } U \ni x, \exists i \in \mathbb{N} \text{ such that } x \in U_i \in U,$$

and show that it is second countable if and only if

$$\exists x \in X, \forall \text{ open sets } \{U_i\} \text{ such that } \forall \text{ open } U \ni x, \exists i \in \mathbb{N} \text{ such that } x \in U_i \in U.$$

Intuitively, first countable spaces have *locally* countable bases around each point and second countable spaces have *globally* countable bases for the whole space.

Second countable spaces are nice as they satisfy an important condition known as *separability*.

Definition 3.9. A topological space is *separable* if it has a countable dense subset.

Proposition 3.10. *If a topological space X is second countable then it is separable.*

Proof. Say that $\{B_i\}_{i\in\mathbb{N}}$ is a countable base for X. Without loss of generality, we may assume that each B_i is non-empty. So, for each $i \in \mathbb{N}$, we may choose $x_i \in B$. If U is a non-empty subset of X, there exists $i \in \mathbb{N}$ such that $B_i \subseteq U$ and hence $x_i \in U$. We conclude from this that the x_i form a countable dense subset. □

Example 3.11. The set of real numbers is separable, indeed, $\mathbb{Q}$ is an appropriate subset. Note that two real numbers may be "separated" by a rational, which goes some way to explaining the terminology, though, before getting too cocky, the reader is referred to definition 4.2 below.

Moreover, it is true that $\mathbb{R}^n$ with the Euclidean topology is separable. In fact, in this case, separability and second countability are actually equivalent. This is a consequence of the more following more general proposition.

Proposition 3.12. *If X be a metric space with the metric topology, then X is second countable if and only if it is separable.*

Proof. We have just seen that any second countable topological space is separable. We must show the converse in the special case that X is metrisable. To that end, assume that X is separable, that is, has a countable dense subset $\{x_i\}_{i\in\mathbb{N}}$. Since X is a metric space, it makes sense to consider the following family of open balls

$$\mathcal{B} := \{B_r(x_i) : r \in \mathbb{Q}^+, i \in \mathbb{N}\}.$$

The family $\mathcal{B}$ is countable as a countable union of countable sets. We will show that it is a base.
Let $x \in X$ and consider a neighbourhood N of X. As N is a neighbourhood, there exists by definition $\varepsilon > 0$ such that $B_\varepsilon(x) \subset N$. As $\{x_i\}$ is dense, there exists $i \in \mathbb{N}$ such that $x_i \in B_{\varepsilon/2}(x)$. By the density of the rationals in the reals, there is $r \in \mathbb{Q}$ such that

$$d(x, x_i) < r < \varepsilon/2.$$

One then deduces from the triangle inequality that

$$x \in B_r(x_i) \subset B_\varepsilon(x) \subset N,$$

as required. □

4. Separation Axioms, Quotient Spaces and Manifolds

Metric spaces are our friends, providing us with a rich source of examples in topology. If only every topological space could be a metric space. Actually, how do we know this isn't the case? How can we be sure that, given a topological space, there isn't some unspecified complicated metric on the underlying set which induces the topological structure. We have a word for that state of affairs.

Definition 4.1. A topological space (X, τ) is *metrisable* if there exists a metric d on X such that the topology on X inhereted from this metric coincides with τ.

One can ask when a topological space is metrisable, though we sadly do not have time to get into it in this course. But we will get into *separation axioms*, which are a route to giving a satisfactory answer to this question - the idea is that topological spaces with the correct separation properties are metrisiable. So, what is a separation axiom? Separability, despite the term, is a countability axiom, and not a separation axiom. There are a *lot* of separation axioms[5], and we will not cover them all in this course. Still, one particular axiom is rather important.

Definition 4.2. A topological space X is *separated* if for all pairs of points $x, y \in X$, there are open neighbourhoods U of x and V of y that are disjoint.

This definition encompasses the basic idea of separation axioms, namely, a specification of how well a topological space allows us to distinguish points. Admittedly, the term "separated" is rather close to "separable", and there is scope for confusion. With that in mind, many resources use the term "Haussdorff" in place of "separated". We will use the two terms interchangably.

By way of contrast to separation, one sometimes wants to identify two points in a topological space via an equivalence relation. That is, we think of two points as "the same" if they lie in the same equivalence class. Such equivalence relations often arise from a *group action* on a set, and this will be of considerable importance in the second half of this course. In order to do this identification thoroughly, we must first show that the *quotient space* has a reasonable topology. With that in mind, recall that, given a topological space X and an equivalence relation $\sim$ one always has the *quotient map*:

$$q : X \to X/\sim := \{\text{equivalence classes of } \sim \text{ in } X\}$$
$$x \mapsto [x], \text{ the equivalence class of } x.$$

Definition 4.3. Let X be a set with an equivalence relation $\sim$. The *quotient topology* on $X/\sim$ is the coarsest topology such that the quotient map $q : X \to X/\sim$ is continuous.

Exercise 4.4. Let $\sim$ be an equivalence relation on a topological space X, with quotient map q. Show that the quotient topology on $X/\sim$ is composed of open sets of the form $\{E : q^{-1}(E) \text{ is open in } X\}$.

Example 4.5. Many interesting examples of quotient spaces appear as quotients of the square $[0, 1]^2 \subset \mathbb{R}^2$. For example, if we identify two opposite sides with the same orientation we produce a *cylinder*. If we make this identification with opposite orientation we obtain a

[5] Have a look on the Wikipedia page for *separation axiom*. The page on *history of separation axioms* may also be of interest.

Mobius strip. Starting from the cylinder, we may identify the two boundary circles with the same orientation and recover a *torus*. With the opposite orientation we construct a *Klein bottle*.

Example 4.6. Yet more interesting examples can be obtained from quotients of vector spaces. Let V be a two-dimensional vector space over $\mathbb{R}$ and define an equivalence relation on $V - \{0\}$ by $v_1 \sim v_2$ iff there is some non-zero $\lambda \in \mathbb{R}^\times$ such that $v_1 = \lambda v_2$. The quotient space $(V - \{0\})/\sim$ is known as a *projective space* $\mathbb{P}(V)$ and can be thought of as the set of lines through the origin in V. Such lines are specified by one paramater, namely the gradient, so this space is considered one-dimensional. By choosing a basis for V we may identify it with $\mathbb{R}^2$ and $\mathbb{P}(V)$ can be denoted $\mathbb{P}^1(\mathbb{R})$ and called the *projective line*. There is an analogous construction for a 3-dimensional vector space, giving rise to the *projective plane*, which is in fact a quotient of the Mobius strip.

By definition, the quotient map is continuous. It is also quite often *open*, especially in geometric situations. It is not, however, a homeomorphism, as it certainly need not be bijective. A general question to ask is how does the topological quotient map interact with other topological properties? For example, if X is a Hausdorff topological space and $\sim$ is an equivalence relation on X, is it true that $X/\sim$ is also Hausdorff? The answer is no, and the student should seek to construct examples of this phenomenon. Still, some quotient spaces are Hausdorff, in particular, those which are *manifolds* such as the cylinder, torus, etc. constructed above.
Manifolds are geometric objects which look locally like Euclidean space. That is, around every point on the manifold there is an open neighbourhood homeomorphic to an open ball in Euclidean space. This idea is made more formal by the notion of *local homeomorphism*.

Definition 4.7. A map $f : X \to Y$ between topological spaces is a local homeomorphism if, for every $x \in X$, there is an open set U containing x such that $f(U)$ is open in Y and the restriction $f|U : U \to f(U)$ is a homeomorphism.

Definition 4.8. A *manifold* is a second-countable Hausdorff topological space which is locally homeomorphic to Euclidean space.

Generally one is interested in manifolds of *fixed dimension*, meaning that the neighbourhood of every point is isomorphic to an open ball in a Euclidean space of fixed dimension. For example, a *surface* is a manifold of dimension 2. Wheras it can be rather difficult to classify topological spaces up to homeomorphism, it turns out that it is possible to do so for *compact* surfaces. We will see a definition of compactness later in this course, but when there is an embedding into Euclidean space it suffices to say that such manifolds are precisely those which are closed and bounded. The first major dividing line in the family of compact surfaces is *orientability*. For example, the Mobius strip is not orientable. In a sense, this means it only has one "side". The orientable compact surfaces are characterised up to homeomorphism by their *genus*, that is, number of "holes". When the genus is 0 we have a sphere, when the genus is 1 we have a torus. To produce a surface of genus 2, one removes a disc from two tori, and glues the remaining spaces along the boundary. This idea can be made more rigorous with the quotient topology, and it can be iterated to construct compact orientable surfaces of higher genus. A similar procedure enables one to construct the compact non-orientable surfaces by gluing copies of the projective plane $\mathbb{P}^2(\mathbb{R})$. For example, one constructs the Klein bottle by gluing two projective planes.

5. Topological Groups

We will now study some interactions of algebra with topology. More specifically, we will be adding topological structure to groups[6]. A group with a satisfactory topological structure is called a *topological group*. By "satisfactory", what we mean is that the group G is given a topology with respect to which, in particular, the group operation defines a *continuous* map $G \times G \to G$. We thus see the need for introducing a natural topology on product spaces.

Definition 5.1. Let $\{X_\alpha : \alpha \in A\}$ be an indexed set of topological spaces, the product topology on $\prod_{\alpha \in A} X_\alpha$ is the coarsest topology such that all of the projections

$$\pi_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta$$
$$(x_\alpha)_{\alpha \in A} \mapsto x_\beta,$$

are continuous.

The formulation above allows for both finite and infinite products, though for topological groups we will only need the former case. The following examples are something of a sanity check.

Example 5.2. Some product spaces, for example $\mathbb{R}^n$, already have a glaring topological structure, which in the case of $\mathbb{R}^n$ is the Euclidean topology. It is true that $\mathbb{R}^n$ with the Euclidean topology is homeomorphic to the n-fold Cartesian product $\mathbb{R} \times \cdots \times \mathbb{R}$ with the product topology.

Example 5.3. We all know that we can consider the real plane $\mathbb{R}^2$ in both Cartesian and polar co-ordinates. We can think of points on the unit circle S^1 as "angles", denoted by θ and we have a homeomorphism

$$f : S^1 \times (0, \infty) \to \mathbb{R}^2 - \{0\}$$
$$f(\theta, r) = (r\cos(\theta), r\sin(\theta)).$$

Example 5.4. Earlier we constructed the torus as a quotient of the square. Alternatively, it can be shown that the torus is homeomorphic to the product of two circles $S^1 \times S^1$.

Definition 5.1 as written above is not very constructive. To that end, the following exercises provide a more workable framework.

Exercise 5.5. Let $X_1, \ldots, X_n$ be a finite family of topological spaces, show that a base for the product topology consists of sets which can be written in the form

$$\pi_1^{-1}(U_1) \cap \cdots \cap \pi_n^{-1}(U_n)$$

for some open $U_i \subset X_{\beta_i}$. Can you write down a basis of the product topology for infinite products?An answer will be given in the following lecture.

Such infinite products will appear again in the next lecture on compactness. Now that we have a satisfactory definition of topology on product spaces we can give the promised definition of topological group.

[6] Contrast this to what happens in section 7, in which we will start with an arbitrary topological space and construct from it a group (the so-called *fundamental group*)

Definition 5.6. A *topological group* is a triple $(G, \tau, \cdot)$ in which (G, τ) is a topological space and $(G, \cdot)$ is a group such that the following conditions hold:

(1) The inverse operation

$$G \to G$$

$$g \mapsto g^{-1}$$

is a continuous map.

(2) The group operation $(g_1, g_2) \mapsto g_1 \cdot g_2$ is a continuous map $G \times G \to G$ when $G \times G$ is endowed with the product topology.

Under the discrete topology, *every group* is a topological group. What is more interesting is when we can give a group a more "natural" topology. But what exactly does that *mean*? Hopefully a few examples should illustrate this somewhat vague concept!

Example 5.7. We have seen lots of topological spaces in these notes that have a clear group structure: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, S^1. Each of these is in fact a topological group with respect to their usual topologies and of these only $\mathbb{Z}$ is discrete.

Example 5.8. A finite group is often thought of as having the discrete topology. In this sense, topological groups with the discrete topology, that is *discrete groups*, are a generalisation of finite groups.

Example 5.9. Let $\mathbb{C}^{n^2}$ have the product topology, and consider $\mathrm{GL}_n(\mathbb{C})$ as a subset with which we endow the subspace topology. $\mathrm{GL}_n(\mathbb{C})$ is then a topological group under matrix multiplication, which is given by polynomial maps. $\mathrm{GL}_n(\mathbb{R})$ has the subspace topology inherited from $\mathbb{R}^{n^2}$. Similarly one has matrix subgroups such as $\mathrm{SL}_2(\mathbb{R})$, which will appear repeatedly in your lectures on hyperbolic geometry. The subgroup $\mathrm{SL}_2(\mathbb{Z})$ inherits the discrete topology and is thus a discrete group.

In fact, this example illustrates a special case of a more general phenomena. Specifically, any subgroup of a topological group is a topological group under the subspace topology. The dedicated reader may like to prove this claim!
We have previously introduced a notion of "equivalence" for topological spaces, namely homeomorphisms. On the other hand, elsewhere you should have seen the analogous notion for groups, that is, isomorphisms. It seems reasonable to combine these two definitions in the context of topological groups.

Definition 5.10. A map $f : G_1 \to G_2$ between topological groups G_1 and G_2 is a topological homomorphism (resp. isomorphism) if it is continuous and a group homomorphism (resp. isomorphism).

Plenty of standard homomorphisms on groups become topological homomorphisms when we add reasonable topological structure to the groups involved. The following example is a case in point.

Example 5.11. Consider the topological group $\mathbb{R}^\times = \mathrm{GL}_1(\mathbb{R})$. The determinant map $\det : GL_n(\mathbb{R}) \to \mathrm{GL}_1(\mathbb{R})$ gives a homomorphism of topological groups. It is surjective, but not injective, and it's kernel is the topological group $\mathrm{SL}_n(\mathbb{R})$.

6. Compactness

As mentioned previously, finite groups are often thought of as discrete topological groups. In this section we introduce a generalisation of finiteness known as *compactness*. Previously we have mentioned one or two limitations on the "size" of a topological space, namely, first countable; second countable and separable. Each of these conditions has to do with cardinality, which is an aspect of set theory, wheras compactness involves more developed aspects of topology. Compact sets are a generalisation of closed and bounded subsets of Euclidean space.

Definition 6.1. Let X be a topological space and $C \subset X$. An open cover of C is a collection $\{U_\alpha\}$ of open subsets of X such that

$$C \subset \cup_\alpha U_\alpha.$$

An (open) subcover is a set $\{U_{\alpha_\beta}\} \subset \{U_\alpha\}$ which also covers C. If a subcover is finite, we say "finite subcover".

The definition of compactness has to do with the cardinality of the indexing sets of open covers.

Definition 6.2. A subset C of a topological space X is *compact* if each of its open covers has a finite subcover.

The classic examples of compact sets are very familiar.

Example 6.3. The theorem of Heine–Borel[7] states that a subset of $\mathbb{R}^n$ (with the Euclidean topology) is compact if and only if it is closed and bounded. From this one concludes that $\mathbb{R}$ is not compact, and nor is the open interval $(0, 1)$ compact. One can use this to deduce compactness of certain matrix groups, eg. $O_n(\mathbb{R})$, $U_n(\mathbb{C})$, Sp_n.

In general, compact subspaces need not be closed unless the space is Hausdorff. A corollary is that a map from a compact space to a Hausdorff space is a homoemorphism if it is a continuous bijection (ie. we need nothing about inverse!).

Example 6.4. As closed and bounded subsets of Euclidean space the circle and the torus are compact spaces. In the case of the circle, this result also follows from the fact that the image of a compact space under a continuous surjective map is compact. For example S^1 is the image of the map $f : [0, 1] \to S^1$, $f(t) = \exp(2\pi i t)$. The fact that the torus is compact may also be deduced from the fact that a product of compact spaces in compact. This result even holds for infinite products, as we will prove below.

Although $S^1, O_n(\mathbb{R})$, etc. are compact topological groups, there are evidently plenty of topological groups which are not compact. Still, many topological groups (especially in this course!) satisfy the following weaker condition.

Definition 6.5. A topological space is *locally compact* if every point has a compact neighbourhood.

Proposition 6.6. *A topological group is locally compact if and only if the identity has a compact neighbourhood.*

[7] Covered in the *metric spaces* course.

Proof. Clearly if a topological group is locally compact then the identity has a compact neighbourhood. For the converse, start by observing that a continuous image of a compact subset is compact. More precisely, let $f : X \to Y$ be a continuous map between topological spaces and let $C \subset X$ be compact, we claim that $f(C)$ is also compact. Indeed, let $\cup_\alpha U_\alpha$ be an open cover of $f(C)$, so that $\cup_\alpha f^{-1}(U_\alpha)$ is an open cover of C. As C is compact, there are finitely many α_i such that

$$C \subset \cup_i f^{-1}(U_{\alpha_i}).$$

It follows that $f(C)$ has the finite subcover $\cup_i U_{\alpha_i}$. Next note that for any element $g \in G$, the map $x \mapsto gx : G \to G$ is a continuous map. It follows that if C is any compact neighbourhood of the identity then gC is a compact neighbourhood of g. □

Example 6.7. $\mathbb{R}$ is a locally compact topological group, for example $[-1, 1] \subset \mathbb{R}$ is a compact neighbourhood of the identity according to the Heine–Borel theorem. Can you prove that $\mathrm{GL}_2(\mathbb{R})$ is a locally compact group? This will be covered in problems class 2.

Locally compact topological groups are especially nice. One reason is that they admit a *Haar measure*, which is a generalisation of the Lebesgue measure on $\mathbb{R}$. Given such a measure, one may develop a theory of integration. It is possible to study a special case of this in one of the year 4 projects. Another nice thing about locally compact groups is their "duality" properties. For example, if a locally compact topological group G is *abelian* then there is a notion of a *dual group* $G^\vee$ such that $(G^\vee)^\vee$. Combining this result with the integration theory hinted at previously, one may develop *harmonic analysis* (Fourier series etc.) on locally compact abelian groups. This is very important, though beyond the scope of this course.

We finish this section by considering products of compact spaces. Tychonoff's theorem says that *any*, finite or infinite, product of compact spaces is compact. We will prove this in the *countable* case.

Theorem 6.8. *Let $X_1, X_2, X_3 \dots$ be topological spaces. If each X_i is compact, then the product $\prod_{i=1}^\infty X_i$ is compact.*

Proof. We argue by contradiction. To that end, suppose we have an open cover $\mathcal{O} = \{U_\alpha\}$ with no finite subcover. Under this assumption, there exists $x_1 \in X_1$ such that no basis set $U_1 \times X_2 \times \dots$ with $x_1 \in U_1$ is covered by finitely many sets in $\mathcal{O}$. Indeed, if, for every $x_1 \in X_1$, there was such a basis set then there would be a finite open subcover of $\mathcal{O}$. Similarly, there is an $x_2 \in X_2$ such that no basis set $U_1 \times U_2 \times X_3 \times \dots$ with $(x_1, x_2) \in U_1 \times U_2$ is covered by finitely many sets in $\mathcal{O}$. In fact, one can choose an infinite sequence of points $x_i \in X_i$ such that, for all $n \in \mathbb{N}$, no basis set $U_1 \times \dots \times U_n \times X_{n+1} \times \dots$ with $(x_1, \dots, x_n) \in U_1 \times \dots \times U_n$ is covered by finitely many sets in $\mathcal{O}$. This is a contradiction because the point $(x_1 m x_2, \dots)$ has to lie in some set in $\mathcal{O}$, which, being open, must contain a basis set $U_1 \times \dots \times U_n \times X_n \times \cdots$ containing $(x_1, x_2, \dots)$, in particular, it is covered by a single set in $\mathcal{O}$. □

Remark 6.9. Note that the proof relied here on the basis of the product topology. It is crucial that the basis of the product topology in the infinite case is given by products of open sets

$$U_1 \times U_2 \times \cdots,$$

such that, for all but finitely many i, $U_i = X_i$. Contrast this to the seemingly natural *box topology*, which has a basis given by arbitrary products of open sets.

7. Connectedness, Homotopy and Fundamental Groups

Discrete groups are *totally disconnected*. This intuitive statement can be given rigorous meaning upon definition of what it means for a topological space to be *connected*. This leads naturally to the defininition of the *fundamental group* of a topological space. In a sense, the fundamental groups measures the connectedness of a topological space by quantifying the obstruction to continuously deforming one *loop* on a space to another. The basic idea is that a space should be regarded as disconnected if it is the union of two or more separate subspaces.

Definition 7.1. A topological space X is *connected* if it cannot be written as the disjoint union of two nonempty open sets[8].

Example 7.2. In problems class 2 we will show that the closed interval $[a, b] \subset \mathbb{R}$ is connected. On the other hand, the rational numbers $\mathbb{Q} \subset \mathbb{R}$ is not a connected topological space. As for matrix groups, for example $\mathrm{SO}_n(\mathbb{R})$ is connected, but $\mathrm{O}_n(\mathbb{R})$ is not.

If a space is not connected then it may be decomposed into the union of a collection of maximal disjoint connected subspaces. The term *maximal* means that none of these connected subspaces is contained in any larger connected subspace. A space with no non-trivial connected subsets is known referred to as *totally disconnected*. We will now march on to the definition of the fundamental group of a topological space. From now on we will use the notation $I := [0, 1]$ for the unit interval.

Definition 7.3. A *path* in a topological space X is a continuous map $f : I \to X$.

The following definition is a notion of equivalence between two paths known as homotopy. Loosely speaking, we say that two paths are *homotopic* if one may be continuously deformed into the other.

Definition 7.4. A *homotopy of paths* in a topological space X is a continuous map

$$H : I \times I \to X$$
$$F(s, t) = f_t(s),$$

such that $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t. If there exists a homotopy between paths we will say that they are *homotopic*.

The notation $f_t(s)$ is intended to suggest that we may think of a homotopy between paths as a continuous family of paths. The independence condition says that in this family, all paths have the same "endpoints".

Example 7.5. A subset $Y \subset \mathbb{R}^n$ is called *convex* if, for any two points $y_1, y_2 \in Y$ the line between them also lies in Y. In this case, any two paths in Y are homotopic. In this case, the line (segment) between two points $y_1, y_2 \in Y$ may be parametrised explicitly and used this to give an explicit homotopy map between any two paths $u, v : I \to Y$.

The student may wish to check that this defines an equivalence relation on the set of paths with fixed endpoints. We want to form a group out of the equivalence classes of certain paths. This means that we must be able to compose two paths, at least when they have a common endpoint.

[8] This is equivalent to saying that the space has no non-trivial clopen subsets.

Definition 7.6. Say u and v are paths in X such that $u(1) = v(0)$, then we may define the *composite path* $u \cdot v$ by

$$u \cdot v(t) = \begin{cases} u(2t) & 0 \leq t \leq \frac{1}{2} \\ v(2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

This means that u and v are traversed twice as fact. If the path u (resp. v) in X be a homotopic to u' (resp, v'), then it can be shown that the composition $u \cdot v$ is homotopic to $u' \cdot v'$. The conclusion to be drawn from this is that composition of paths respects homotopy, and we can define composition on the level of equivalence classes. This is what we desire for our group! The issue is how we guarantee that one path ends where the next begins? A sensible answer is to deal only with *loops*, which we now formally define.

Definition 7.7. A *loop based at a point* $b \in X$ is a path $l : I \to X$ such that $l(0) = l(1) = b$.

Evidently, we may compose two loops based at the same point $b \in X$. The fundamental group builds a group operation from such compositions.

Definition 7.8. Let X be a topological space with a fixed base point b. The *fundamental group* $\pi_1(X, b)$ is defined as the set of homotopy classes of loops based at b with the group operation induced from composition of paths.

Is this a reasonable definition? More specifically, does this actually define a group? We need to check the axioms! Firstly, is the operation even well-defined? If so, is composition of loops associative? Actually, it only needs to be associative up to homotopy, that is, given loops t, u and v based at b, is $t(uv)$ homotopic to $(tu)v$? What is the identity of this group operation? Given a homotopy class of a based loop, what is its inverse?

Lemma 7.9. *Let X be a topological space and let $b \in X$ be a base point. $\pi_1(X, b)$ is a group under path composition of homotopy classes of loops.*

Proof. See Hatcher's *Algebraic Topology* section 1.1, which is freely available online. $\square$

The fundamental group might seem rather abstract and impractical to compute, but it can be done sometimes! For example, let $X \subseteq \mathbb{R}^n$ be a convex set with basepoint $x_0 \in X$. The fundamental group $\pi_1(X, x_0)$ is the trivial group, due to the fact that any two loops based at x_0 are homotopic via a linear homotopy. This illustrates the notion of *simply connected* topological spaces, which are those with trivial fundamental group, such that, for each pair of points $a, b \in X$, there exists a path in X from a to b. The existence of paths between all pairs of points in X is often referred to as *path-connected*. It can be shown that every path-connected space is connected. On the other hand, there are examples of topological spaces which are connected but not path-connected[9]. The fundamental group, of course, need not be trivial, for example the fundamental group of the circle turns out to be $\mathbb{Z}$, with the integer 1 corresponding to a loop going once round the circle, and -1 corresponding to its reversal. Knowing the fundamental group of the circle has some surprising applications. An example of this is the *fundamental theorem of algebra*. All of this and more is covered in the module Algebraic Topology. That said, many spaces do have a trivial fundamental group, including even generalisations of the circle, such as the 3-sphere S^3. On the other hand, the Poincare conjecture[10] states that every simply connected closed 3-manifold is homeomorphic to S^3.

[9]For example, you may wish to google the *topologist's sine curve*.

[10]which is now a theorem!