QUATERNION ALGEBRAS AND MODULAR FORMS

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We wish to know about generalizations of modular forms using quaternion algebras. We begin with preliminaries as follows.

1. A few preliminaries on quaternion algebras

As we must to make a correct statement in full generality, we begin with a profoundly unhelpful definition.

Definition 1. *A* quaternion algebra *over a field k is a 4 dimensional vector space over k with a multiplication action which turns it into a central simple algebra*

Four dimensional vector spaces should be somewhat familiar, but what of the rest? Let's start with the basics.

Definition 2. *An* algebra *B over a ring R is an R-module with an associative multiplication law(hence a ring).*

The most commonly used examples of such rings in arithmetic geometry are affine polynomial rings $R[x_1, \ldots, x_n]/I$ where R is a commutative ring and I an ideal. We can have many more examples though.

Example 1. *If R is a ring (possibly non-commutative), $n \in \mathbf{Z}_{\geq 1}$ then the ring of n by n matrices over R(henceforth, $M_n(R)$) form an R-algebra.*

Definition 3. *A* simple ring *is a ring whose only 2-sided ideals are itself and (0)*

Equivalently, a ring B is simple if for any ring R and any nonzero ring homomorphism $\phi : B \to R$ is injective. We show here that if $R = k$ and $B = M_n(k)$ then B is simple.

Suppose I is a 2-sided ideal of B. In particular, it is a right ideal, so $BI = I$. For any $i \in I$, $b \in B$, the j-th column on bi is b times the j-th column of i. Furthermore, if the j-th column of i is zero, then the j-th column of bi is zero. Meanwhile, if the j-th column of i is nonzero, say $(i_1, \ldots, i_n)^\top$ and without loss of generality, $i_1 \neq 0$, then we can find the j-th column of bi to be anything, say $(a_1, \ldots, a_n)^\top$ by

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taking b so that all but the first column is zero and the first column is $(a_1/i_1, \ldots, a_n/i_n)^\top$. Therefore, we cannot control the content of the j-th column unless all j-th columns are zero in I and so any right ideal must be of the form I_S where $S \subset \{1, \ldots, n\}$ and I_S is the set of matrices whose j-th column is zero if $j \in S$. Likewise, I is also a left ideal, so $IB = I$. Therefore it must also be of the form I^S where S is of the same form and the rows in the S indices are zero.

Then say we want the 2-sided ideal where the j-th row is zero. Then since it is a left ideal, we can only mandate that the j-th row is zero by letting all columns which intersect the j-th row to be zero, that is, all columns must be zero. Hence there are only two 2-sided ideals, (0) and the entire algebra.

Another important example of a simple algebra over a field is a division algebra. Division algebras are algebras where any nonzero element is invertible. Contrast this to the case of matrix algebras where it is very easy to find non-invertible elements (just find zero-divisors!). We now give an important theorem on simple algebras:

Theorem 1. (*Wedderburn*)*Let A be a simple algebra over k. Then $A \cong M_n(D)$ where D is a division algebra over k.*

We omit a proof here(it is in Dummit & Foote as Theorem 18.2.4), but the application to quaternion algebras is clear.

Corollary 1. *Let H be a quaternion algebra over k. Then either $H \cong M_2(k)$ or $H \cong D$ where D is a 4-dimensional division algebra.*

We give give the last definition with no further ado:

Definition 4. *We say a k-algebra A is* central *if the center, $Z(A) = k$. That is, if $\alpha \in A$ and for all $\beta \in A$, $\alpha\beta = \beta\alpha$ then $\alpha \in k$.*

It is an elementary fact from linear algebra that matrix rings are simple algebras ($Z(M_n(k))$ are scalar diagonal matrices). By more calculation we may also prove that the center of Hamilton's Quaternions is $\mathbf{R}$.

Finally we give a theorem, not for any particular use other than culture:

Theorem 2. (*Skolem-Noether*) *Any automorphism of a Central Simple Algebra is given by conjugation.*

Proof. [MCFT, Theorem IV.2.10] □

2. Quaternion Algebras

Let B be a quaternion algebra over a field k. Thus $B \cong kx_0 \oplus kx_1 \oplus kx_2 \oplus kx_3$ as a k-vector space($x_i \neq 0$, else B is not a 4-dimensional vector space).

If $B \cong M_2(k)$ then we can take $x_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and the other choices are obvious. As we will see later, this is the case of the classical modular curve and the case where B is a division algebra/k is much more interesting so we consider that to be the case.

Since $x_i \neq 0$, we may, up to left multiplication by x_0^{-1} take $x_0 = 1$.

We wish now to show that $B = (a, b)_k$ for some $a, b \in k^\times$ (provided the characteristic is not 2, as we will see soon).

Consider that $x_1, x_2, x_3 \notin k$, else the dimension is not 4. Since x_1 is invertible, $k(x_1)$ is a field. What degree is it over k? Consider that since $1, x_1, x_1^2, x_1^3, x_1^4$ all lie in B there must be a linear dependence between them, hence x_1 is a root of a monic degree 4 polynomial $f \in k[X]$. If f is irreducible, then $k[X]/(f)$ is a degree 4 vector space lying inside of B, so $B = k[X]/(f)$. This is however in contradiction to the definition of B as a central simple algebra (in that case the center of B would be B). If f has a degree 3 factor, then it must have a degree 1 factor, and hence $x_1 \in k$. Hence f must split into the product of 2 irreducible monic quadratics. Hence there exist $\alpha, \beta \in k$ such that $x_1^2 = \alpha x_1 - \beta$. Then as long as $char(k) \neq 2$, we can complete the square and add an element of k to x_1 to get and element i such that $i^2 = a \in k^\times$. For an elementary look at quaternion algebras in characteristic 2, see problem set 4 in [Co04].

Then consider the map $f : B \to B$ by $b \mapsto ibi^{-1}$. Since f^2 is the identity and f is a linear map on k-vector spaces, B decomposes uniquely into eigenspaces for ± 1. Pick some element from the -1 eigenspace and call it j. Since $ij = -ji$, multiplying this equation on the right gives $ij^2 = -jij = (-j)(-j)i = j^2 i$ so j^2 is in the $+1$ eigenspace. By the same logic as above, j satisfies a monic quadratic polynomial, so $j^2 = cj + b$. We already know cj is in the -1 eigenspace while $j^2 - b$ is in the $+1$ eigenspace so $cj = j^2 - b$ must be zero, i.e. $j^2 = b$. It is clear now (by dimension counting!) that $k(i)$ is the $+1$ eigenspace while $k(i)j$ is the -1 eigenspace so $B = k(i) \oplus k(i)j = k \oplus ki \oplus kj \oplus kij$.

Now that we have the representation of B by $(a, b)_k$, we introduce the canonical involution on B, where if $b = x + iy + jz + ijw$ then $\overline{b} = x - iy - jz - ijw$. Note that $b = \overline{b}$ if and only if $b \in k$. Thus

$t(b) = b + \bar{b}$, $n(b) = b\bar{b}$ are in k. Moreover, $\chi_b(X) = (X - b)(X - \bar{b}) = X^2 - t(b)X + n(b)$ is the minimal polynomial for b.

Note that $M_2(k)$ is a quaternion algebra for any field k. Consider the representation $(a,b)_k$ of any quaternion algebra only matters up to squares in $k^\times$. Suppose then if a is a square, we consider the map given in [AB04] $(1,b)_k \to M_2(k)$ by

$$x + iy + zj + wij \mapsto \begin{pmatrix} x+y & z+w \\ b(z-w) & x-y \end{pmatrix},$$

with inverse

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \frac{1}{2}\Big((\alpha+\delta) + (\alpha-\delta)i + (\beta + \frac{\gamma}{b})j + (\beta - \frac{\gamma}{b})ij\Big).$$

This is useful as if $k \subset F$ then if we were to consider $(a,b)_F = (a,b)_k \otimes_k F$ then (for instance if a, b or ab were squares in F) the structure of our quaternion algebra may change. In particular if $(a,b)_F \cong M_2(F)$ then we say F splits the quaternion algebra $(a,b)_k$.

We will now consider a very specific set of field extensions, the completions of a field with respect to its nontrivial absolute values (which we call *places*). Since we are most interested in the case where $k = \mathbf{Q}$ we make that restriction here, as there is little change in the theory to a general field with an infinite number of places and characteristic $\neq 2$, but we restrict to a familiar case for simplicity. Again,

Hereon, $k = \mathbf{Q}$ and $(a,b) := (a,b)_{\mathbf{Q}}$.

It is a theorem of Ostrowski that the nonarchimedian places of $\mathbf{Q}$ are indexed by the primes ideals of $\mathbf{Z}$. Moreover there is only one archimedian metric on $\mathbf{Q}$, which we denote ∞. For any metric v, we will have Cauchy sequences (x_n), the set of Cauchy sequences with respect to that metric is denoted $\mathcal{C}(v)$. With some proof we can show that these form a ring with maximal ideal $\mathcal{N}(v)$ made up of the sequences which tend to zero. We denote $\mathbf{Q}_v = \mathcal{C}(v)/\mathcal{N}(v)$.

Definition 5. *We define the Hilbert Symbol for a quaternion algebra (a,b) and a place v as*

$$(a,b)_v := \begin{cases} 1 & (a,b)_{\mathbf{Q}_v} \cong M_2(\mathbf{Q}_v) \\ -1 & \textit{else} \end{cases}.$$

Moreover, if $(a,b)_v = 1$ we say (a,b) splits *at v and otherwise we say that it is* ramified.

There is a well-known formula about Hilbert Symbols which is now pertinent, so we state but do not prove it:

Theorem 3. (*Hilbert Product Formula*):

$$\prod_v (a,b)_v = 1$$

As a corollary, (a,b) ramifies at a finite, even number of places. Interesting is that we can use global class field theory ([MCFT, Theorem VIII.6.13]) to see that an even number of places determines a quaternion algebra (a,b) ramified at exactly those places. We collect this information as follows:

Definition 6. *We say that a quaternion algebra (a,b) is* definite *if $(a,b)_\infty = -1$ and* indefinite *otherwise. An indefinite quaternion algebra is then determined by an even number of primes in $\mathbf{Z}$ and a definite quaternion algebra is determined by an odd number of primes in $\mathbf{Z}$. In either case, we call the* reduced discriminant *of a quaternion algebra the squarefree product of those primes.*

3. Orders in Quaternion Algebras(over $\mathbf{Q}$)

Important in algebraic number theory is the idea of a subring of a field called the maximal order which captures the essence of that field. We wish to have a corresponding definition for a quaternion algebra.

The first thought might be to consider the set of all integral elements of (a,b). We do indeed have a corresponding definition as every element α of a quaternion algebra has a minimal polynomial $X^2 - t(\alpha)X + n(\alpha)$ and we do call α *integral* when $t(\alpha), n(\alpha) \in \mathbf{Z}$, which is a useful characterization for local fields, but is insufficient for $\mathbf{Q}$. For one thing, the set of integral elements of a quaternion algebra will not generally be a ring. Consider the example for $M_2(\mathbf{Q})$ of [Cl03]

$$A = \begin{pmatrix} 1/2 & -3 \\ 1/4 & 1/2 \end{pmatrix}, B = \begin{pmatrix} 0 & 1/5 \\ 5 & 0 \end{pmatrix}.$$

Note that both A and B are integral, but neither of AB nor $A+B$ are integral.

We step back from that precipice then and call an *order* of a quaternion algebra B to be a subring $\mathcal{O}$ of B which contains $\mathbf{Z}$, whose elements are all integral and whose tensor from $\mathbf{Z}$ to $\mathbf{Q}$ is B. Likewise we call a *maximal order* to be one which is maximal with respect to inclusion. We can show one exists because the set of orders is nonempty for any quaternion algebra (a,b). Just take $\mathcal{O} = \mathbf{Z} \oplus i\mathbf{Z} \oplus j\mathbf{Z} \oplus ij\mathbf{Z}$ and we can use Zorn's lemma on the set of orders of B containing $\mathcal{O}$(or on any other given order).

By the property that $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Q} \cong B$, we must have $\mathcal{O}$ be a rank 4 lattice. We define the discriminant $D_{\mathcal{O}}$ of $\mathcal{O}$ to be the unique positive integer such that if as a $\mathbf{Z}$-lattice, $\mathcal{O} \cong \langle x_1, x_2, x_3, x_4 \rangle$ then $D_{\mathcal{O}}^2 = \det(t(x_i x_j))$. Note that this is well defined, because if we also have $\mathcal{O} \cong \langle x_1', x_2', x_3', x_4' \rangle$ then for all i, $x_i' = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4$ and $x_i = b_{i1}x_1' + b_{i2}x_2' + b_{i3}x_3' + b_{i4}x_4'$ with the coefficients in $\mathbf{Z}$. If we let $A = (a_{ij})$ and $B = (b_{ij})$ then BA is the 4×4 identity matrix so $A, B \in GL_4(\mathbf{Z})$.

Then if we let

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ \\ \overline{x_1} & \overline{x_2} & \overline{x_3} & \overline{x_4} \end{pmatrix}, M' = \begin{pmatrix} x_1' & x_2' & x_3' & x_4' \\ \\ \overline{x_1'} & \overline{x_2'} & \overline{x_3'} & \overline{x_4'} \end{pmatrix},$$

then $M' = MA$ and the matrix $(t(x_i x_j)) = M^\top M$ while $(t(x_i' x_j') = (MA)^\top MA = A^\top (M^\top M) A$. Thus $\det(t(x_i' x_j')) = \det(A)^2 \det(M^\top M) = (\pm 1)^2 \det(t(x_i x_j)) = \det(t(x_i x_j))$.

This realization above gives a useful way to tell the index of an order in a maximal order because if $\mathcal{O} \cong \langle x_1', x_2', x_3', x_4' \rangle$ is an order of index N inside of a maximal order $\langle x_1, x_2, x_3, x_4 \rangle$ and M denotes the change of basis matrix $x_i \to x_i'$ then by the theory of lattices, $N = |\det(M)|$. It follows that if $\mathcal{O}$ is not maximal, it has the same index in any maximal order containing it.

An interesting fact is that the discriminant of an order is $D_{\mathcal{O}} = DN$ where D is the reduced discriminant of B and N is the index of the order. In particular, an order is maximal if and only if its discriminant is D. This is one way to show that $M_2(\mathbf{Z})$ is a maximal order in $M_2(\mathbf{Q})$.

Definition 7. *Let $\mathcal{O}$ be given as the intersection of two maximal orders so that its index in either is N. Then we call $\mathcal{O}$ an* Eichler order *of level N.*

Example 2. *The set of matrices*

$$\mathcal{O}_0(1, N) := \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} : a, b, c, d \in \mathbf{Z} \right\}$$

is a level N order in $M_2(\mathbf{Z})$.

We conclude the section with a proposition about Eichler orders and Local Fields(Found in [AB04] as Proposition 1.53 parts (a) and (c)).

Theorem 4. *Let $\mathcal{O}$ be an order in a quaternion algebra over $\mathbf{Q}$ with discriminant D and let N be a positive integer. Then the following are equivalent statements:*

- *$\mathcal{O}$ is an Eichler order of level N*
- *If p is a prime then $\mathcal{O}_p = \mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is maximal if $p|D$ and is isomorphic to $\begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ N\mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix}$ otherwise.*

The proof of this theorem relies heavily on the fact that up to conjugation, quaternion algebras over $\mathbf{Q}_p$ have a unique maximal order $M = \{b \in B | n(b) \in \mathbf{Z}_p\}$. More specifically, up to conjugation, any order of $M_2(\mathbf{Q}_p)$ can be written as $\begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ p^n\mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix}$.

4. Modular Forms and Quaternion Algebras

Hereon, all quaternion algebras are indefinite.

We do this because the ultimate aim of this note is to explore generalizations of modular forms given by quaternion algebras. The emphasis then is on generating discrete subgroups of $SL_2(\mathbf{R})$ using quaternion algebras. We have a ready set of $\mathbf{Z}$-lattices with a multiplicative structure on them in the orders, and if we can realize those inside of $M_2(\mathbf{R})$ we have a great chance of finding something useful. Certainly if we require that $B \otimes_{\mathbf{Q}} \mathbf{R} \cong M_2(\mathbf{R})$ then $\mathcal{O} \hookrightarrow B \hookrightarrow M_2(\mathbf{R})$ as a discrete subgroup.

We note the following useful fact:

Lemma 1. *The reduced norm of an element in $M_2(\mathbf{R})$ is its matrix norm and likewise for the reduced trace.*

Proof. Ignoring anything about reduced trace or reduced norm, a 2×2 matrix A satisfies the polynomial $X^2 - T(A)X + \det(A)$ after a short calculation. Since minimal polynomials are monic, they are unique, so $T(A) = t(A)$ and $\det(A) = n(A)$. Note this shows the analogue to the standard involution for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the classical adjoint $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. □

Thus we consider $\mathcal{O}^\times$ and indeed its subgroup $\mathcal{O}_1^\times$ of elements of reduced norm 1. Note that if $\mathcal{O} = M_2(\mathbf{Z})$ then $\mathcal{O}_1^\times$ is the classical modular group $SL_2(\mathbf{Z})$. If $\mathcal{O} = \mathcal{O}_0(1, N)$ then $\mathcal{O}_1^\times = \Gamma_0(N)$.

There is a corresponding general construction for an analogue of the congruence subgroups defined as follows: let $\mathcal{O}$ be a maximal order in a quaternion algebra B with reduced discriminant D and let N be a positive integer coprime to D. Then if $N = \prod_{i=1}^g p_i^{e_i}$ then $\mathcal{O} \hookrightarrow B \hookrightarrow$

$B \otimes_{\mathbf{Q}} \mathbf{Q}_{p_i} \cong M_2(\mathbf{Q}_{p_i})$. Since $\mathcal{O}$ is made up of integral elements, it must (up to conjugation) map into $M_2(\mathbf{Z}_{p_i})$, so we have a map $\mathcal{O} \hookrightarrow M_2(\mathbf{Z}_{p_i}) \twoheadrightarrow M_2(\mathbf{Z}/p_i^{e_i}\mathbf{Z})$.

Thus by the chinese remainder theorem, we have a map $\mathcal{O} \rightarrow M_2(\mathbf{Z}/N\mathbf{Z})$ and by the theorem from Alsina, the matrices which are upper-triangular under this map form an Eichler order of level N in $\mathcal{O}$.

Thus, given an order $\mathcal{O}$ in a quaternion algebra, let $\mathcal{O}_+ = \mathcal{O}_1^\times / \pm 1$ acts properly discontinuously on the upper half-plane as it is discrete in $SL_2(\mathbf{R})$. Thus we have a discrete group and can start talking about quotients by the upper-half plane.

Theorem 5. *Let $\mathcal{O}$ be an order in a quaternion algebra B of discriminant $D > 1$. Then the corresponding* Shimura Curve $X(\mathcal{O}) = \mathcal{O}_+ \setminus \mathcal{H}$ *is compact*

We've already proven much of this in class. Miyake [My89] shows how a discrete subgroup Γ of $SL_2(\mathbf{R})$'s quotient by $\mathcal{H}$ is compact after adding a finite number of cusps, which correspond to nontrivial parabolic subgroups of Γ. Up to conjugation it further shows that a parabolic element is (when considered in $SL_2(\mathbf{R})$) of the form $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$.

Now suppose such an element α lay in $\mathcal{O}_1^\times$. Then $\alpha - 1 \in B$, and in $M_2(\mathbf{R})$ this looks like $\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix}$ and is thus nilpotent. This is a contradiction though, as if $D > 1$ then B is a division algebra, and thus has no zero-divisors. Therefore $\mathcal{O}_+$ has no parabolic subgroups, so there are no cusps and $X(\mathcal{O})$ was already compact.

Note that since there are no cusps, any modular form we could define would trivially be a cusp form. Moreover, since there are no parabolic elements there are no Fourier expansions to be found on any possible modular forms.

Definition 8. *Let $\mathcal{O}_1^\times$ be as above. If under the embedding into $SL_2(\mathbf{R})$, $\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix}$, we say a* modular form *of weight k on $\mathcal{O}$ is a holomorphic function $f : \mathcal{H} \rightarrow \mathbf{C}$ such that $f(\alpha z) = (c_\alpha z + d_\alpha)^k f(z)$.*

Note that as in the classical case, cusp forms(now all modular forms) of weight $2n$ give holomorphic differentials on $X(\mathcal{O})$.

Remark 1. *It turns out that indefinite rational quaternion algebras satisfy "The Eichler Condition" which guarantees that maximal (and thus level N Eichler) orders are all conjugate. Thus $X(\mathcal{O})$ is actually independent of the order $\mathcal{O}$, and only depends on the level. As such, if*

$\mathcal{O}$ is an order of level N, $X(\mathcal{O})$ is often called $X_0^D(N)$ and if the level is 1, this is often shortened to just X^D.

5. The modular interpretation

5.1. **The world's shortest introduction to abelian varieties.** As we are working with quotients of the upper-half plane, we consider only abelian varieties over $\mathbf{C}$. We could give the following unhelpful definition:

Definition 9. *An abelian variety A is a complete, connected group variety.*

As this is not a geometry class, we don't use this definition. Instead we consider that over $\mathbf{C}$, we have a differentiable map from the tangent space (isomorphic to $\mathbf{C}^{\dim A}$ to A which is in this case onto. The kernel of the exponential map(the name of this map) is a full lattice in $\mathbf{C}^{\dim A}$ (isomorphic to $\mathbf{Z}^{2\dim(A)}$). One can alternately show that a complex vector space V mod a full lattice L has the structure of an abelian variety if and only if L admits a "Riemann Form"(that is, we have an embedding into projective space, making it a complete variety).

Definition 10. *A Riemann Form is a skew-symmetric bilinear form on L*

$$E : L \times L \to \mathbf{Z}$$

such that the following holds (note that if we consider that $V = L \otimes_{\mathbf{Z}} \mathbf{R}$, then $E_{\mathbf{R}} : V \times V \to \mathbf{R}$)

- $E_{\mathbf{R}}(iv, iw) = E_{\mathbf{R}}(v, w)$
- *The associated Hermitian form $H(v, w) = E_{\mathbf{R}}(iv, w) + iE_{\mathbf{R}}(v, w)$ is positive definite.*

Thus the definition for an abelian variety is the one we give as follows:

Definition 11. *An abelian variety of dimension g is a complex vector space V of dimension g together with a full lattice $L \subset V$ which admits a Riemann form E.*

For an example, an elliptic curve is an abelian variety of dimension 1. We consider abelian varieties of dimension 2, i.e. abelian surfaces.

Remark 2. *There is one final piece to the puzzle for abelian varieties, called polarizations. We ignore these, as Hermitian forms naturally give polarizations and in our case there is a canonical Riemann form as we will shortly see.*

5.2. **Abelian surfaces and Quaternionic Multiplication.** This section will have the purpose of showing the following:

Theorem 6. *If $\mathcal{O}$ is a maximal order in a quaternion algebra B then there is a bijection between $\mathcal{O}_+\backslash\mathcal{H}$ and the set of abelian surfaces $A/\mathbf{C}$ with multiplication action by $\mathcal{O}$.*

If $\mathcal{O}_N$ is a level N Eichler order, then there is a bijection between $(\mathcal{O}_N)_+\backslash\mathcal{H}$ and the set of abelian surfaces $A/\mathbf{C}$ with multiplication by $\mathcal{O}$ decorated by a cyclic $\mathcal{O}$ submodule of A with group structure isomorphic to $\mathbf{Z}/N\mathbf{Z}\oplus\mathbf{Z}/N\mathbf{Z}$.

Consider the following uniformization map:

Let $\tau\in\mathcal{H}$ and let $\mathcal{O}_N$ be a level N order in a quaternion algebra B and $\mathcal{O}$ the maximal order containing $\mathcal{O}_N$. Consider that $\mathcal{O}\hookrightarrow M_2(\mathbf{R})\hookrightarrow M_2(\mathbf{C})=End(\mathbf{C}^2)$. Thus $\mathcal{O}\binom{\tau}{1}$ is a full lattice in $\mathbf{C}^2$ with an evident $\mathcal{O}$-action. We will show that $\mathbf{C}^2/\mathcal{O}\binom{\tau}{1}$ is an abelian surface of a special type, but to do that we need a Riemann Form. If $x,y\in\mathcal{O}$, then we let $E(x\binom{\tau}{1},y\binom{\tau}{1}))=t(\alpha x\overline{y})$ where α is a canonical choice of an element of B with trace zero. This is bilinear by the linearity of the trace and is alternating because $x\overline{x}=n(x)\in\mathbf{Q}$ so $\alpha n(x)$ has trace zero.

Consider that we do indeed have a canonical choice of α from our quaternion algebra B.

Theorem 7. (*Hasse's Criterion, Latimer's version for* $\mathbf{Q}$) *A quadratic extension F of $\mathbf{Q}$ embeds into the indefinite rational quaternion algebra of discriminant D if and only if p is ramified or inert for all $p|D$.*

Thus $\mathbf{Q}(\alpha)\hookrightarrow B$ by Hasse's criterion if $\alpha^2+D=0$, which guarantees p ramified for all $p|D$. So each $\tau\in\mathcal{H}$ defines an abelian surface A_τ. Now compare the surfaces defined by τ and $\gamma\tau$ where $\gamma\in\mathcal{O}_1^\times$.

Recall first that if $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ in $SL_2(\mathbf{R})$ then $\gamma\tau=(a\tau+b)/(c\tau+d)$ as the action on $\mathcal{H}$ is induced from the action of $SL_2(\mathbf{R})$ on $\mathbb{P}^1(\mathbf{C})$. Thus

$$\mathcal{O}\begin{pmatrix}\gamma\tau\\ 1\end{pmatrix}=\mathcal{O}\begin{pmatrix}(a\tau+b)/(c\tau+d)\\ 1\end{pmatrix}=\mathcal{O}\begin{pmatrix}a\tau+b\\ c\tau+d\end{pmatrix}=\mathcal{O}\gamma\begin{pmatrix}\tau\\ 1\end{pmatrix}=\mathcal{O}\begin{pmatrix}\tau\\ 1\end{pmatrix}.$$

If $\mathcal{O}_N=\mathcal{O}$ is a maximal order and $N=1$, we have thus defined our bijection between $\mathcal{O}_+\backslash\mathcal{H}$ and quaternionic abelian surfaces. If $\mathcal{O}_N$ is not maximal, then $\mathcal{O}_N\binom{\tau}{1}$ is an abelian surface with multiplication by $\mathcal{O}_N$. Alternately, let G_N be a subgroup of A_τ in the sort described in the statement of the theorem. Then the set of $\gamma\in\mathcal{O}$ such that $\gamma G_N\subset G_N$ forms an Eichler order of level N (details on this can be found in [Cl03]).

6. The Hecke Algebra and Quaternionic Eichler-Shimura

Unfortunately, the moduli problem which a ramified quaternion algebra parametrizes does not seem to afford such an easy description of a Hecke operator as a sum over lattices, as most rank 4 lattices do not give an abelian variety, let alone one with quaternionic multiplication. We now reexamine the Hecke algebra in the style of Shimura. Recall first the k-slash operator for functions meromorphic on $\mathcal{H}$. If $\sigma \in GL_2^+(\mathbf{R})$,

$$f|_k\sigma(\tau) := \det \sigma^{k/2}(c_\sigma\tau + d_\sigma)^{-k} f(\sigma\tau).$$

Shimura additionally proved that if Γ_1, Γ_2 are *commensurable*(their intersection is of finite index in each) subgroups of $GL_2^+(\mathbf{R})$ and α belongs to *the commensurator* of Γ_1,

$$\{\alpha \in \Gamma_1 : \alpha\Gamma\alpha^{-1} \text{ commensurable with } \Gamma\},$$

then $\Gamma_1\alpha\Gamma_2$ decomposes as a finite union of disjoint cosets $\bigcup \Gamma_1\alpha_i$. Using this we can define a k-slash operator on double cosets:

$$f|_k\Gamma_1\alpha\Gamma_2 := \det \alpha^{k/2-1} \sum f|_k\alpha_i,$$

which preserves $A_k(\Gamma_1), M_k(\Gamma_1), S_k(\Gamma_1)$[Sh70].

It can then be shown (as in [DS05]) that our standard Hecke operator T_p is given by a double coset where $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ and $\Gamma_1 = \Gamma_2 = \Gamma_\bullet(N)$.

We cannot give a direct analogue of this construction for a general quaternion algebra, as we see, due to Hasse's criterion.

Example 3. *Consider $X(\mathcal{O})$ where $\mathcal{O}$ is an order in the quaternion algebra of discriminant 6.. There is not even an element of reduced norm 2 in the quaternion algebra B of discriminant 6. If there were, then $\mathbf{Q}(\sqrt{-2})$ would embed into B. However, 3 splits into distinct primes in $\mathbf{Q}(\sqrt{-2})$, hence it cannot embed into B.*

What then can we do? Well much as we did to find an analogue of $\Gamma_0(N)$, we look everywhere p-adically. To define T_{p^k} for $p \nmid D$(for $p|D$ we have a division algebra, and while every quadratic extension splits it, we no longer have a canonical choice) we simply take the action of the double coset $(\mathcal{O}_1^\times \otimes \mathbf{Z}_p) \begin{pmatrix} 1 & 0 \\ 0 & p^k \end{pmatrix} (\mathcal{O}_1^\times \otimes \mathbf{Z}_p)$ and consider the action to be trivial at each other place. Clearly this action is commutative for the places where we have defined it, and we consider its action on the space of modular (cusp) forms of weight k.

6.1. **Newforms and Modularity.** Exactly as in the case of classical modular forms, we have a Petersson inner product for $f, g \in S_k(\mathcal{O}_+)$,

$$\int_{X(\mathcal{O})} f(z)\overline{g(z)}\Im(z)^{k-2}\mathrm{d}z.$$

Consider the orders $B \supset R \supset \mathcal{O}$. Since $\mathcal{O}_+ \subset R_+$, any modular form for R is a modular form for $\mathcal{O}$ so $S_k(R_+) \subset S_k(\mathcal{O}_+)$.

Definition 12. *Let $S_k(\mathcal{O}_+)^{new}$ denote the orthogonal complement of $\sum_{R \supsetneq \mathcal{O}} S_k(R_+)$ with respect to the Petersson inner product. We denote a* newform *of $\mathcal{O}_+$ to be a modular form in $S_k(\mathcal{O}_+)^{new}$ which is a simultaneous eigenform for the algebra generated by all the $T(p^k)$ for $p \nmid N$.*

The algebra generated is denoted $\mathbb{T}_{\mathbf{Z}}$ and is referred to as the Hecke algebra.

We then have the following theorem, found in [Hi77] as Lemma 1.6.i.

Theorem 8. *As a complex vector space, $S_k(\mathcal{O}_+)$ has a direct sum decomposition into $\bigoplus_{R \supset \mathcal{O}} S_k(R_+)^{new}$.*

Now suppose we have a newform f. By definition if $T \in \mathbb{T}_{\mathbf{Z}}$ then $T(f) = \alpha_f(T)f$ so we have a map $\mathbb{T}_{\mathbf{Z}} \to \mathbf{C}$. We would like to consider the ring $\mathbf{Z}_f = \alpha_f(\mathbb{T}_{\mathbf{Z}})$. If I_f is the kernel of α_f then we have the ring homomorphism $\mathbb{T}_{\mathbf{Z}}/I_f \to \mathbf{Z}_f$.

Consider now $S_2(\mathcal{O}_+)$. As we know, it is isomorphic to $\Gamma(X(\mathcal{O}), \Omega)$, the complex vector space of holomorphic differentials on $X(\mathcal{O})$. Recall that the Jacobian of a complex curve C is defined as $\Gamma(C, \Omega)^\vee/H_1(C, \mathbf{Z})$. In this case

$$J(\mathcal{O}) = \Gamma(X(\mathcal{O}), \Omega)^\vee/H_1(X(\mathcal{O}), \mathbf{Z}) \cong S_2(\mathcal{O}_+)^\vee/H_1(X(\mathcal{O}), \mathbf{Z}).$$

Since H_1 is a free $\mathbf{Z}$-module of rank $2g$, this is a complex torus(and thus an abelian variety, when we consider the intersection pairing as a Riemann form). This also affords us another step as each Hecke operator gives an endomorphism of the abelian variety. Since the Jacobian is a quotient of a g-dimensional vector space, $\mathbf{Z}_f$ must have rank $\leq g^2$.

Now consider I_f as a subalgebra of the Hecke algebra acting on $J(\mathcal{O})$. The image of the action, $I_f(J(\mathcal{O}))$ forms a subvariety of $J(\mathcal{O})$ and we denote the quotient variety $A_f := J(\mathcal{O})/I_f(J(\mathcal{O}))$. The same methods as [DS05, Proposition 6.6.4] give that this a quotient variety of dimension equal to the $\mathbf{Z}$-rank of $\mathbf{Z}_f$.

What is this dimension then? What sort of abelian variety do we get?

Theorem 9 (Zhang, 2001)**.** *If f is a newform of weight 2 for $\mathcal{O}$ then the eigensubspace of $S_2(\mathcal{O}_+)$ of $\mathbb{T}_{\mathbf{Z}}$ with character α_f has dimension 1.*

Corollary 2. *A_f is an abelian variety of dimension 1 over $\mathbf{C}$, i.e. a complex elliptic curve.*

This is perhaps surprising, given that our Shimura Curves parametrize things which may or may not be moduli spaces of elliptic curves but nonetheless they possess a modularity map to an elliptic curve. Thus we find that quaternionic modular forms allow us to discover that although modular curves parametrize elliptic curves with some extra structure, that parametrization is NOT what gives us modularity.

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