TORIC VARIETIES PRESENTATION

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These notes are strongly based off of the short paper "Geometric Invariant Theory and Toric Varieties" by Nick Proudfoot and the book "Lectures on Invariant Theory" by Igor Dolgachev.

1. Jim's part of the presentation

1.1. **When exactly can we take a quotient anyways?** Consider a familiar example. Consider the action of $\mathbb{C}^\times$ on $\mathbb{C}^{n+1}$ by $t \cdot (x_0, \dots x_n) = (tx_0, \dots, tx_n)$.

The orbits of this action are

$$[x_0 : \cdots : x_n] = \{(y_0, \dots, y_n) : y_i = tx_i \text{ for some } t \in \mathbb{C}^\times\}$$

for $(x_0, \dots, x_n)$ such that some $x_i \neq 0$, and the singleton orbit $\{(0, \dots, 0)\}$.

The quotient $\mathbb{C}^{n+1}/\mathbb{C}^\times$ is unfortunately not a nice geometric object. Note for instance that $(0, \dots, 0)$ is a limit of every other point.

Worse yet, the functions on this quotient are exactly the same as those on the point $(0, \dots, 0)$. Suppose that F is a $\mathbb{C}^\times$ invariant function on an open set of $\mathbb{C}^{n+1}$ containing a point $x \in \mathbb{C}^{n+1}$. Let $O(x)$ be the $\mathbb{C}^\times$ orbit of x and note that if y lies in the closure of $O(x)$, then $\mathbb{C}^\times$-invariance demands that $F(x) = F(y)$. But now note that $(0, \dots, 0)$ lies in every orbit closure, so for all $x \in \mathbb{C}^{n+1}$, $F(x) = F((0, \dots, 0))$. Thus as an algebraic object, we may as well just be dealing with the point $(0, \dots, 0)$.

However, if we remove the *unstable point* $(0, \dots, 0)$ and THEN take the quotient, we get our old friend $\mathbb{P}^n$.

Suppose in general that we have a group G acting on a variety X. A good quotient X/G should include a regular map $f : X \to X/G$ so the fibers of this map ought to be closed in X. As shown above, this need not be the case. The solution we saw above is to take a G-invariant Zariski open subset U of X and consider the quotient map $U \to U/G$. Geometric Invariant Theory (GIT) is a method for choosing this subset.

Before we even get started, we restrict to reductive groups so that we can take advantage of the following theorem:

Nagata's Theorem If G is a reductive group acting on a finitely generated k-algebra R then R^G is also finitely generated.

A proof of this fact is found in Dolgachev. We don't define a reductive group here, but we remark that all tori over $\mathbb{C}$ split as $(\mathbb{C}^\times)^n$ so they are semi-simple and thus reductive (whatever that means). Other examples of reductive groups are GL_n and SL_n. Examples of non-reductive groups are $\mathbb{G}_a$ or any group that can be embedded into the upper-triangular matrices with 1's on the diagonal.

1.2. **A review of Projective varieties and line bundles.**

Definition 1.1. Let $S = \bigoplus_{d\in\mathbb{Z}_{\geq 0}} S_d$ be a $\mathbb{Z}_{\geq 0}$-graded ring. We may define a special ideal called the *irrelevant ideal* $S_+ = \bigoplus_{d>0} S_d$. Then $\mathrm{Proj}(S)$ is defined to be the space of *homogeneous* prime ideals $\mathfrak{p} \not\supset S_+$ of S endowed with the Zariski topology.

If you wish, you may think of Proj as Spec but with some extra technical conditions (many people do!). The point however is that the Proj construction gives a line bundle on $\mathrm{Proj}(S)$ called $\mathcal{O}(1)$. It is so named because if S is a $\mathbb{C}$-algebra its global sections are exactly the $\mathbb{C}$ vector space S_1.

Why do we care about line bundles? Well recall first that a line bundle $L \xrightarrow{\pi} X$ is a morphism such that for all $x \in X$ there is an open set U such that $\pi^{-1}(U_x) \cong U_x \times \mathbb{A}^1$. The reason we care about line bundles is that the only regular functions on a projective variety are constants, and therefore, we have no way of embedding an abstract projective variety into projective space using regular functions on that variety. So, instead of regular functions, we use sections of a line bundle over X to embed X into a projective space.

Note that $\mathrm{Proj}(S)$ a projective variety if and only if $\mathcal{O}(1)$ is ample. If G acts on $\mathrm{Proj}(S)$, giving an action of G on S is equivalent to giving an action of G on a line bundle L (which we take to be $\mathcal{O}(1)$). More generally, we call the extension of this action a *linearization* of $X \times G \to X$ to $(X, L) \times G \to (X, L)$.

Once we've defined that action, we have an analogous action on $L^{\otimes m}$ for $m \in \mathbb{Z}$. Now suppose s is a G-invariant global section of $L^{\otimes m}$ for such an m.

Definition 1.2. Consider the set $\{x \in X : s(x) \neq 0\}$. If this is an affine subset of X, we denote it X_s. If $x \in X$ lies in X_s for some G-invariant global section of $L^{\otimes m}$ then we say that x is *semistable*. If not, we say that x is *unstable*. We denote the points which are semistable with respect to the linearization on L by $X^{ss}(L)$.

Example 1.3. Consider $X = \mathbb{C}^{n+1}$ and $G = \mathbb{C}^\times$ under the action we described at the start. It can be shown that every line bundle is isomorphic to the trivial bundle $X \times \mathbb{A}^1$. Thus a linearization will be defined by a character $\chi : \mathbb{C}^\times \to \mathbb{C}^\times$ so that $t \cdot (x, v) = (tx, \chi(t)v)$. Of course such a character must be of the form $t \mapsto t^m$ for some integer m.

A global section of the trivial line bundle on $\mathbb{C}^{n+1} = \mathrm{Spec}(\mathbb{C}[t_0, \ldots, t_n])$ will be given by a polynomial $F(t_0, \ldots, t_n)$ so that $s(x) = (x, F(x)$. The action of t on s via the linearization above is given as $(t \cdot s)(x) = (x, t^m F(t^{-1}x))$.

Thus the G-invariant sections are given by the F which are homogeneous polynomials of degree m.

If $m < 0$ there are no G-invariant sections so there are no semistable points. Hence $X^{ss}(L)/G = \emptyset$.

If $m = 0$, the only G-invariant sections are constant, so the whole space is semistable. However the only regular functions will be the G-invariant sections, so the quotient is just a point.

However, if $m > 0$ we have a linearization that will actually help us. If $x = (x_0, \ldots, x_n) \neq (0, \ldots, 0)$ then there exists some $x_i \neq 0$. This means that there is a G-invariant section s for which $s(x)$ is nonzero (t_i^m) and for which X_s is affine. Meanwhile $s((0, \ldots, 0)) = ((0, \ldots, 0), 0)$ for every G-invariant section s. Therefore, this is the only unstable point. $\bullet$

Remark 1.4. Essentially the same argument can be made to show that if S is a graded ring, $\mathrm{Proj}(S)$ is the *GIT* quotient of $\mathrm{Spec}(S)$ by $\mathbb{C}^\times$.

Definition 1.5. Define the GIT quotient $X^{ss}(L)/G$ of X, L by G as $X//_L G$.

In the above example, we could take any $m > 0$, but if $m > 1$ we could recognize the linearization as equivalent to the linearization on the m-th tensor power of the trivial bundle. In general, it can be difficult to find a suitable linearization, but at least in the case of varieties and schemes which are projective over affine, we can say the following:

Theorem 1.6. $\mathrm{Proj}(S)//_{\mathcal{O}(1)}G = \mathrm{Proj}(S^G)$. *Moreover, if* $\mathrm{Proj}(S)$ *is projective (that is,* $\mathcal{O}(1)$ *is ample) then* $\mathrm{Proj}(S^G)$ *also is.*

2. Maxim's part of the presentation

Fix an n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of integers and let $(\mathbb{C}^*)^n$ act on $R = \mathbb{C}[x_1, \dots, x_n, t]$ by the rules

$$\lambda \cdot x_i = \lambda_i x_i \text{ and } \lambda \cdot t = \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n} t$$

for $\lambda \in (\mathbb{C}^*)^n$. This can be viewed as an action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^n$ with a linearization to the trivial bundle given by α. Note that $\mathbb{C}^n = \mathrm{Proj}\,\mathbb{C}[x_1, \dots, x_n, t]$, where $\deg x_i = 0$ and $\deg t = 1$. Indeed, $\mathrm{Proj}\,\mathbb{C}[x_1, \dots, x_n, t]$ can be defined as the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the action of $\mathbb{C}^*$, acting trivially on the first n coordinates and acting by scalar multiplication on the last coordinate.

Let $G \leq (\mathbb{C}^*)^n$ be an algebraic subgroup and consider the short exact sequence of algebraic groups

$$1 \to G \to (\mathbb{C}^*)^n \to T \to 1$$

and the induced short exact sequence of tangent spaces at the identity elements

$$0 \to \mathfrak{g} \to \mathfrak{t}^n \to \mathfrak{t} \to 0,$$

which is a short exact sequence of complex Lie algebras. Let $\{e_1, \dots, e_n\}$ be the coordinate vectors in $\mathfrak{t}^n \simeq \mathbb{C}^n$ and let $\bar{e}_i$ be the image of e_i in $\mathfrak{t}$. We have the lattice

$$\mathfrak{t}_\mathbb{Z} = \ker\{\exp : \mathfrak{t} \to T\} \subset \mathfrak{t}$$

and we define $\mathfrak{t}_\mathbb{R} = \mathfrak{t}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}$. Given an n-tuple of integers $\alpha = (\alpha_1, \dots, \alpha_n)$ as before, we may define a polyhedron

$$\Delta = \{v \in \mathfrak{t}_\mathbb{R}^* \,|\, v \cdot \bar{e}_i \geq \alpha_i \text{ for all i }\}$$

and let

$$\sigma = \sigma_\Delta = \mathrm{Closure}\big(\{(v, r) \in \mathfrak{t}_\mathbb{R}^* \times \mathbb{R} \,|\, r \geq 0 \text{ and } v \in r \cdot \Delta\}\big)$$

be the cone over Δ. (Draw a cone over the interval, a cone over a half-line and a cone over a triangle.) Let

$$S_\sigma = \sigma_\Delta \cap (\mathfrak{t}_\mathbb{Z}^* \times \mathbb{Z})$$

be the semigroup of all the lattice points in the cone σ.

Theorem 1. $R^G \simeq \mathbb{C}[S_\sigma]$. Geometrically, the quotient $\mathbb{C}^n//G$ is isomorphic to the toric variety $\mathrm{Proj}\,\mathbb{C}[S_\sigma]$.

Example 2.1. Let $\Delta = \mathbb{R}_{\geq 0} \subset \mathbb{R}$ be the ray of non-negative real numbers. Then $\sigma = (\mathbb{R}_{\geq 0})^2$ and $\mathbb{C}[S_\sigma] = \mathbb{C}[x, t]$ with $\deg x = 0$ and $\deg t = 1$. The associated toric variety is $\operatorname{Proj} \mathbb{C}[x, t] \simeq \mathbb{C}$. Geometrically, $T \simeq \mathbb{C}^*$ and G is the trivial group $\{1\}$. So, we took the quotient of $\mathbb{C}$ by the trivial group and got $\mathbb{C}$ back. More generally, the positive orthant in $\mathbb{R}^d$ corresponds to the toric variety $\mathbb{C}^d$, equipped with the trivial line bundle.

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Example 2.2. Let $\Delta = [0, 1]$ be the closed unit interval in $\mathbb{R}$. Then $S_\sigma = \mathbb{C}[x, y]$ is a polynomial ring with x, y having degree 1. The associated toric variety is $\mathbb{P}^1$. Geometrically, $\mathbb{C}^*$ acts on $\mathbb{C}^2$ by coordinate-wise scalar multiplication and the origin is the unique unstable point. More generally, the standard d-simplex in $\mathbb{R}^d$ gives rise to the toric variety $\mathbb{P}^d$ with the standard line bundle $\mathcal{O}_{\mathbb{P}^d}(1)$.

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Example 2.3. Let $\Delta = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. We may check that

$$\mathbb{C}[S_\sigma] = \mathbb{C}[x, y, z, w]/(xz - yw),$$

where x, y, z, w each have degree 1. Geometrically, we have an action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^4$ given by

$$(\lambda, \mu) \cdot (z_1, z_2, z_3, z_4) = (\lambda z_1, \lambda z_2, \mu z_3, \mu z_4).$$

The unstable locus consists of points where either $z_1 = z_2 = 0$ or $z_3 = z_4 = 0$. The quotient of semistable points by $(\mathbb{C}^*)^2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

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